MA5Q6 GRADUATE ALGEBRA - HOMEWORK 6

DUE TUESDAY 20/11, 12PM

Hand in the first five questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(1) Let A, B, C be R-modules. Show that $\mathrm{Hom}(A \oplus B, C) \cong \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C)$ and $\mathrm{Hom}(A, B \oplus C) \cong \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, C)$. "Hom commutes with direct sum".

(2) Prove the five lemma: Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"] & B \arrow[r, "g"] \arrow[d, "m"] & C \arrow[r, "h"] \arrow[d, "n"] & D \arrow[r, "f"] \arrow[d, "p"] & E \arrow[d, "q"] \\ A' \arrow[r, "r"] & B' \arrow[r, "s"] & C' \arrow[r, "t"] & D' \arrow[r, "u"] & E' \end{tikzcd}$$

be a commutative diagram with exact rows. Show that if m and p are isomorphisms, l is a surjection, and q is an injection, then n is an isomorphism.

(3) We showed in lecture that if

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

is a short exact sequence of R-modules and either $\mu : C \to B$ is an R-module homomorphism with $\phi \circ \mu = \mathrm{id}_C$ or $\lambda : B \to A$ is an R-module homomorphism with $\lambda \circ \psi = \mathrm{id}_A$, then the sequence splits, so $B \cong A \oplus C$. What is the situation if we instead have a short exact sequence of arbitrary (not necessarily abelian) groups? Hint: In one case you find that B is a product, and in another just a semidirect product.

(4) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. Hint: Show that if F is a free $\mathbb{Z}$-module, then $\bigcap_{n \in \mathbb{N}} (nF) = \{0\}$.

(5) Show that if P_1, P_2 are projective modules then $P_1 \oplus P_2$ are projective modules.

(6) (Not to be handed in). Which conjecture of Serre did Quillen and Suslin prove?

(7) (Not to be handed in, and only for those with geometric interest). What do projective modules have to do with vector bundles?

(8) (Not to be handed in, and only for those who like categories). What extra requirements do we need on a category so that the homological algebra we've discussed for R-modules still makes sense?