MA5Q6 GRADUATE ALGEBRA - HOMEWORK 5

DUE TUESDAY 13/11, 12PM

Hand in the first six questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(1) Show that if $S \cong \oplus_{k=0}^{\infty}$ is a graded ring and I is an ideal of S with $I = \langle I \cap S_i : i \geq 0 \rangle$ (ie I is generated by the homogeneous elements in I), then S/I is a graded ring.
(2) Show that assigning $\deg(x) = 2$ and $\deg(y) = 3$ gives the polynomial ring $S = K[x, y]$ the structure of a graded ring (ie describe the homogeneous pieces S_i, and show that $S \cong \oplus S_i$, and $S_i S_j \subseteq S_{i+j}$.
(3) Show that if K, L are fields with $K \subset L$, then $L \otimes_K K[x_1, \ldots, x_n] \cong L[x_1, \ldots, x_n]$ as L-algebras.
(4) Let R be a commutative ring with unit, and let M be an R-module. Show that if A is a commutative R-algebra and $\phi : M \to A$ is an R-module homomorphism, then there is a unique R-algebra homomorphism $\Phi : S(M) \to A$ such that $\Phi|_M = \phi$.
(5) Show that the map that takes M to $T(M)$ is functorial (so given $f : M \to M'$ we get an induced $T(f) : T(M) \to T(M'))$.
(6) Let V be a two-dimensional vector space over a field K, with basis $\mathbf{e}_1, \mathbf{e}_2$. Describe $\wedge V$ as a vector space over K (ie give a basis). Given a linear map $\phi : V \to V$, describe the induced map $\phi : \wedge V \to \wedge V$ in this basis.
(7) (Not to be handed in). Generalize what what you noticed for $\wedge^2 V$ in the previous exercise to arbitrary finite dimensional vector spaces.