MA5Q6 GRADUATE ALGEBRA - HOMEWORK 4

DUE TUESDAY 6/11, 12PM

Hand in the first five questions and one of Q6/Q7/Q8 to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(1) Let $m, n \in \mathbb{N}_{>0}$. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

(2) Prove directly that if A, B are right R-modules and C is a left R-module, then $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$.

(3) Show that if R is a commutative ring, and M, N are R-modules (which we regard as both left and right R-modules), then $M \otimes_R N \cong N \otimes_R M$, where the isomorphism is as R-modules.

(4) Let A be a finitely generated abelian group of rank r (so $A \cong \mathbb{Z}^r \oplus A'$ where A' is a finite abelian group). Show that $A \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r$. (This question assumes that you know the classification of finitely generated abelian groups. Let me know if not).

(5) Let $\phi : V \to V'$ and $\psi : W \to W'$ be linear transformations of finite-dimensional vectors spaces over a field k. Let A be the matrix of ϕ with respect to a particular choice of bases for V and V', and let B the matrix of ψ with respect to a choice of bases for W and W'. Describe the matrix for $\phi \otimes \psi : V \otimes_k W \to V' \otimes_k W'$.

(6) Recall that a poset (partially ordered set) P can be regarded as a category, whose objects are the elements of P, and for which $hom(A, B)$ is a single element if $A \leq B$ and empty otherwise.
   (a) If P and Q are posets, what is a functor $F : P \to Q$? (ie describe it in poset-language).
   (b) What does it mean for two functors F, G to be adjoint?

(7) Show that there is no functor G : **Sets** $\to$ **Groups** which is *right* adjoint to the forgetful functor F : **Groups** $\to$ **Sets** (Hint: coproducts).

(8) Let F be the forgetful functor from the category **Top** of topological spaces to the category **Set** of sets that takes a topological space to its underlying set. Show that F has both a left and a right adjoint (ie there exist functors G, H : **Set** $\to$ **Top** such that $\hom(GX, Y) \cong \hom(X, FY)$ and $\hom(Y, HX) \cong \hom(FY, X)$ for all sets X and topological spaces Y.

(9) (Not to be handed in) What is a Dynkin diagram? Why are they important?