MA5Q6 GRADUATE ALGEBRA - HOMEWORK 3

DUE TUESDAY 30/10, 12PM

Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(1) Let X be a set with $|X| = n$. Show that the abelianization of $F(X)$ is isomorphic to $\mathbb{Z}^n$. Conclude that if $|X| < |Y| < \infty$ then $F(X) \ncong F(Y)$.
(2) The free product of $G = (X|R)$ and $H = (Y|R)$ is the group $G * H = (X \sqcup Y|R \cup R')$. Show that $G * H$ is the coproduct of G and H in the category of groups. Deduce that this construction did not depend on the choice of presentation of G and H.
(3) Describe the coproduct in the category of groups in general.
(4) Describe explicitly the coproduct of $\mathbb{Z}/2\mathbb{Z}$ with itself in the category of groups and in the category of abelian groups. Why does the fact that they are different not contradict the uniqueness of coproduct?
(5) Let $G = (X|R)$ be a group given by generators and relations. Show that if $f : X \to H$ is function from X to a group H with the property that for any word $x_1^{\delta_1} \dots x_n^{\delta_n}$ in R we have $f(x_1)^{\delta_1} \dots f(x_n)^{\delta_n} = e$ then there exists a unique group homomorphism $\bar{f} : G \to H$ with $\bar{f}(x) = f(x)$.
(6) Let G be the group $(x_1, \dots, x_n | x_i^2$ for $1 \leq i \leq n, x_i x_j x_i^{-1} x_j^{-1}$ if $|i - j| \geq 2, (x_i x_{i+1})^3$ for $1 \leq i \leq n-1)$. Which familar group is this?
(7) (Not to be handed in) Let G be the group $(x_1, \dots, x_n | x_i x_j x_i^{-1} x_j^{-1}$ if $|i - j| \geq 2, x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ for $1 \leq i \leq n-1)$. This is the *braid group* on $n+1$ strands. What is this? What is the relationship with the group of the previous exercise?
(8) (Not to be handed in) Read more about the word problem for groups.
(9) (Not to be handed in, and only for those who want more category theory). What is Yoneda's lemma?
(10) (Not to be handed in) Read Gowers blogpost on "it can easily be seen that": `http://gowers.wordpress.com/2008/09/17/princeton-companion-errata/`