MATH 559 HOMEWORK 6

DUE: WEDNESDAY, APRIL 25

All rings R are commutative with 1, and if not otherwise noted M and N are R-modules. Warning: I don't have the most recent printing of Eisenbud - if the "name" of an exercise doesn't coincide with its number, please let me know immediately. For Eisenbud "graded" means $\mathbb{Z}$-graded unless otherwise stated.

(1) (Repeated, augmented, question). Let R be a $\mathbb{Z}$-graded ring with R_0 a field. Many things that are true for local rings are also true for R.
  (a) Let M be a graded R-module, and let $\mathfrak{m} = R_{>0}$ be the unique maximal homogeneous ideal. Then $M = 0$ if and only if $M_{\mathfrak{m}} = 0$. (You may assume that R is Noetherian here).
  (b) (Graded Nakayama). Let M be a graded R-module, and let I be a homogeneous ideal generated by elements of positive degree. Then if $IM = M$ we have $M = 0$.
  (c) If M and N are graded R-modules with $M \otimes_R N = 0$, then $M = 0$ or $N = 0$.

(2) Since the completion of the local ring R_m at m_m is equal to the completion of R at m, and $R \subseteq \hat{R}_m$ when R is Noetherian, we know that the localization of $\mathbb{Z}$ at $p\mathbb{Z}$ (fractions with denominators not divisible by p) is contained in $\hat{\mathbb{Z}}_p$. Show this directly by describing $a/b \in \hat{\mathbb{Z}}_p$ where $\gcd(a, b) = 1$ and p does not divide b.

(3) Give a criterion for $a \in \hat{\mathbb{Z}}_p$ to be a cube.

(4) Write out the next three iterates of applying Newton's method to compute $\sqrt{8}$ in $\hat{\mathbb{Z}}_7$ starting with $a_0 = 1$.

(5) Recall that the m-adic topology on $\hat{R}_m$ has basic opens $\{a + \hat{m}_i : a \in \hat{R}_m, i > 0\}$.
  (a) Show that $\hat{R}_m$ is Hausdorff with this topology (so limits are unique).
  (b) Verify that a polynomial $f \in R[x]$ is a continuous function from $\hat{R}_m$ to $\hat{R}_m$ in this topology.