# MATH 559 HOMEWORK 5

DUE: WEDNESDAY, APRIL 11

All rings $R$ are commutative with 1, and if not otherwise noted $M$ and $N$ are $R$-modules. Warning: I don't have the most recent printing of Eisenbud - if the "name" of an exercise doesn't coincide with its number, please let me know immediately. For Eisenbud "graded" means $\mathbb{Z}$-graded unless otherwise stated.

(1) Let $R = k[x, y]$ and let $I = \langle x^2, xy, y^2 \rangle$. Describe the blow-up algebra $R[It]$ by generators and relations.

(2) **Hilbert function and polynomial**. Let $S = k[x_1, \dots, x_n]$ have the standard grading $\deg(x_i) = 1$, and let $I$ be a homogeneous ideal in $R$. The Hilbert function $H_{S/I}(t)$ is the dimension $\dim_k((S/I)_t)$ of the $t$th graded piece of $S/I$.

   (a) Show that $H_{S/I}(t)$ equals $H_{S/\operatorname{in}(I)}(t)$ for any initial ideal $\operatorname{in}(I)$ of $I$. This means that we can reduce computing Hilbert functions of ideals to computing Hilbert functions of monomial ideals.

   (b) Give a formula for the Hilbert function $H_S(t)$ of the polynomial ring $S$. Note that this is a polynomial in $t$.

   (c) If $0 \to S/I \to S/J \to S/K \to 0$ is a short exact sequence, how are the function $H_{S/I}$, $H_{S/J}$, and $H_{S/K}$ related?

   (d) Give an algorithm to compute the Hilbert function for a monomial ideal. Hint: Consider the short exact sequence $0 \to S/(I : x) \to S/I \to S/(I, x) \to 0$, where $x$ is a variable.

   (e) Deduce that $H_{S/I}(t)$ agrees with a polynomial $P_{S/I}(t)$ for sufficiently large $t$. This is called the *Hilbert polynomial* of $S/I$.

   (f) Compute the Hilbert function and polynomial for $I = \langle x^2y, xy^2 \rangle \subset k[x, y]$, and for $J = \langle x^2y - 3yz^2 + 7z^3 \rangle \subset k[x, y, z]$.

(3) Let $R = k[x, y, z]$, and let $M = R/\langle x, y, z \rangle$, and $N = R/\langle x^3 + 7xz^2 - z^3, xyz \rangle$. Compute $\operatorname{Tor}_i(M, N)$ for all $i \geq 0$. You will need to use Macaulay 2, but do not use the Tor command. Illustrate that $\operatorname{Tor}_i(M, N) = \operatorname{Tor}_i(N, M)$ in this case directly from a choice of free resolution for each module.

(4) Write a oral exam question on localization, and one on primary decomposition, and answerit. A good question will be one that can be answered in real time, but illustrates some general principle (for example, asking for an example that shows that a particular theorem is sharp).