MATH 559 HOMEWORK 3

DUE: MONDAY, MARCH 5

All rings R are commutative with 1, and if not otherwise noted M and N are R-modules. Warning: I don't have the most recent printing of Eisenbud - if the "name" of an exercise doesn't coincide with its number, please let me know immediately.

(1) Eisenbud 3.6, 3.7, 3.8. Characterize monomial ideals that are prime, irreducible, radical and primary. Give an algorithm to compute an irreducible decomposition of a monomial ideal, and one to compute a minimal primary decomposition. Illustrate your algorithms with the monomial ideal $I = \langle a^4c, abc^4, a^5b, b^3c^4, a^3c^4, ab^4c^3\rangle \subseteq k[a, b, c, d]$.

(2) Eisenbud 3.10 a-c. Uniqueness of primary decomposition.

(3) Eisenbud 3.17 and 3.18. Prime avoidance.

(4) Find a primary decomposition of the ideal $I = \langle a^2c^2 + 6abc^2 + 5b^2c^2, 6a^3b + 31a^2b^2 - 25b^4, a^4 - 26a^2b^2 + 25b^4\rangle \subseteq \mathbb{C}[a, b, c]$. Hint: You'll probably want to use a computer package. Here are some useful Macaulay 2 commands:

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i2 : I=ideal(a^3,a*b^2,b^5)


                3     2   5
o2 = ideal (a , a*b , b )

i3 : I:b
                     3   4
o3 = ideal (a*b, a , b )
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(5) Show that if I is an ideal in R, and $(I : f^n) = (I : f^{n+1})$, where $(I : g) = \{h \in R : gh \in I\}$, then $I = (I, f^n) \cap (I : f)$.

(6) Let $K = \mathbb{Q}(\sqrt{(d)})$ be the extension field of $\mathbb{Q}$ obtained by adding the square root of d for some $d \in \mathbb{N}$. Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}$ in K. We may assume that d is squarefree (not divisible by the square of any prime). Show that $\{a + b\sqrt{d} : a, b \in \mathbb{Z}\} \subseteq \mathcal{O}$ if $d = 0, 2, 3 \mod 4$, and $\{a + b((1 + \sqrt{d})/2) : a, b \in \mathbb{Z}\} \subseteq \mathcal{O}$ if $d = 1 \mod 4$. Bonus: These are actually equalities (feel free to look up a proof).