MATH 559 HOMEWORK 2

DUE: MONDAY, FEBRUARY 19

All rings R are commutative with 1, and if not otherwise noted M and N are R-modules. Warning: I don't have the most recent printing of Eisenbud - if the "name" of an exercise doesn't coincide with its number, please let me know immediately.

(1) We set $\operatorname{Spec}(R)$ to be the set of prime ideals in R. The *Zariski topology* on $\operatorname{Spec}(R)$ is given by setting the closed sets to be $V(I) = \{P \in R : I \subseteq P, P \text{ is prime }\}$ for an ideal I of R.
   (a) Show that the Zariski topology is a topology. How does it relate to the example of varieties done in class?
   (b) Show that if $\phi : R \to S$ is a ring homomorphism, then we get an induced map from $\operatorname{Spec}(S)$ to $\operatorname{Spec}(R)$.

(2) (a) Show that $M \otimes_R R \cong R \otimes_R M \cong M$.
   (b) Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$.
   (c) Show that if S is an R-algebra, then $S \otimes_R R[x_1, \ldots, x_n] \cong S[x_1, \ldots, x_n]$.
   (d) Show that $k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m] \cong k[x_1, \ldots, x_n, y_1, \ldots, y_m]$.
   (e) Show that if R, S and T are rings, and $\phi : R \to S$, $\psi : S \to T$ are ring homomorphisms (making S into an R-algebra and T into an S-algebra, then if M is an R-module then $(M \otimes_R S) \otimes_S T \cong M \otimes_R T$, where T is an R-algebra via the homomorphism $\psi \circ \phi$.

(3) In class we showed that localization has the following universal property: If $\phi\colon R \to S$ is a ring homomorphism which takes all elements of $U \subset R$ to units of S, then there is a unique induced ring homomorphism from $R[U^{-1}]$ to S. Show that this property defines the localization up to unique isomorphism; if T is a ring for which every ring homomorphism from R to a ring S that takes elements of U to units of S factors uniquely through T, then T is uniquely isomorphic to $R[U^{-1}]$.

(4) Eisenbud Exercise 2.3 (How to localize without admitting it).

(5) Eisenbud Exercise 2.6 (Generalized Chinese Remainder Theorem)

(6) Eisenbud Exercise 2.20 (about localizing at various f_i).

(7) Show that if $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is a radical ideal, then I is prime if and only if there do not exist ideals $J, K \neq I$ with $I = J \cap K$.

(8) Let G be a finitely generated abelian group. What is $\operatorname{Ass}_{\mathbb{Z}}(G)$?