MATH 555 - COMBINATORIAL COMMUTATIVE ALGEBRA

HOMEWORK 2

Routine Calculations

(1) Compute a free resolution using Schreyer's algorithm for the $S = k[x, y, z]$ module $S/\langle x^2, y^2, xy + yz\rangle$. If your answer is not a minimal resolution, minimalize it.
(2) Compute the Taylor resolution for $k[x, y]/I$, where $I = \langle x^{10}, x^5y, x^4y^5, xy^6\rangle$. Minimalize this resolution to get the minimal resolution.
(3) Compute the hull resolution for the previous example.
(4) Let $I = \langle x_1x_3, x_1x_4, x_1x_6, x_2x_6, x_3x_5, x_3x_6, x_4x_6\rangle \subseteq k[x_1, x_2, x_3, x_4, x_5, x_6]$. Compute $\beta_{2,\mathbf{1}}(I)$, where $\mathbf{1} = (1, 1, 1, 1, 1, 1)$.

Gaps from class

(1) Check Stanley's trick to compute the h-vector from the f-vector works. A reference for the trick is Ziegler, p250.
(2) Check that

$$\sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} \binom{n}{i} = \binom{n-d+j-1}{j}.$$

There are undoubtedly many ways to do this - if you're feeling uninspired try playing with generating functions. If you don't know how to do that, be sure to ask me.
(3) (If you don't want to take this on faith) Check that $Tor_i(M, N) = Tor_i(N, M)$. An outline is given as Exercise 1.12 of Miller-Sturmfels.

Broader questions

(1) Minimal free resolutions can depend on the characteristic of k, but Hilbert functions do not, even though we can compute the Hilbert function from the free resolution. Why is this not a problem?
(2) Show that if I is a monomial ideal the Betti numbers $\beta_{0,\mathbf{a}}(I)$, $\beta_{1,\mathbf{a}}(I)$, and $\beta_{n,\mathbf{a}}(I)$ do not depend on the field k.

(3) Compute a primary decomposition for the knights ideal. What does this say about inclusion-maximal configurations of knights on the chess-board?

(4) **The Set Problem**. Let $\mathbb{F}_3$ be the field with 3 elements. Three points (vectors) $x, y, z \in \mathbb{F}_3^n$ form a line if $x + y + z = 0$. The *Set Problem* asks for the maximal size of a set of points in $\mathbb{F}_3^n$ containing no lines. I'm taking the name from the card game Set - these are also called *caps* in coding theory. This is open for $n \geq 6$.

Let S be the polynomial ring with 3^n variables, where each variable corresponds to an element of $\mathbb{F}_3^n$. Let I be the ideal generated by monomials $x_u x_v x_w$, where $u + v + w = 0$.

(a) Show that $\dim(S/I)$ is the answer to the Set Problem.

(b) Compute a free resolution for S/I when $n = 2$.

(c) (Almost completely open, and probably hard!) What can we say about minimal free resolutions of S/I for general n? For example, can we compute the hull resolution? Can we exploit the fact that the group of affine transformations of $\mathbb{F}_3^n$ acts on S/I?