# MATH 555 - COMBINATORIAL COMMUTATIVE ALGEBRA

HOMEWORK 1

## Routine Calculations

(1) Compute $H_{S/M}(20)$ for $M = \langle a^2bc, ad^2e, cde\rangle$ using a different tree of choices of variable than the one given in class, and check that you get the same answer.
(2) What is the Hilbert polynomial of $I = \langle a^3b^2, ac^4, bc^3\rangle$?
(3) Find an irredundant minimal primary decomposition for $I = \langle a^3b^2, ac^4, bc^3\rangle$.

## Gaps from class

(1) Show that $I$ is a monomial ideal if and only if for every $f \in I$ every term of $f$ lies in $I$.
(2) Show that the minimal monomial generators of a monomial ideal are unique.
(3) Show that the characterization of irreducible monomial ideals given in class is correct (ie show that if $I$ is generated by powers of the variables, then $I \neq J \cap K$ for $J$, $K$ properly containing $I$ but not necessarily monomial).
(4) Recall that every associated prime of a monomial ideal is monomial, so if $M$ is a monomial ideal and $f \in S$, then if $M : f$ is prime it must be monomial. If $M : f$ is not prime, must it still be a monomial ideal?

## Broader questions

(1) When does $IJ = I \cap J$ for $I$ and $J$ monomial ideals?
(2) How many bishops can be placed on a $5 \times 5$ chess-board with no two attacking each other? (This is an easier question than the knight one to do directly. I really mean "mimic the algebraic proof from the first class for knights").
(3) Given a monomial ideal $M$ in a polynomial ring in $n$ variables generated by monomials of degree at most $d$, give a bound on the number $N$ such that the Hilbert function $H_{S/M}(n)$ agrees with the Hilbert polynomial for $n > N$.

(4) Generalize our proof that the Hilbert polynomial exists to modules. Hints: Every finitely generated $S$-module is the quotient of a free $S$-module. For Gröbner reasons you may assume that the quotient is by a module generated by "monomials". What is the corresponding short exact sequence here?

(5) Show that there is a canonical primary decomposition of a monomial ideal in the following sense: Every monomial ideal has a unique minimal primary decomposition $I = \cap Q_\sigma$ for which each $Q_\sigma$ is a monomial ideal that is $P_\sigma$-primary, and $Q_\sigma$ is maximal among all possible $P_\sigma$-primary components. (Eisenbud, exercise 3.11, attributed to Bayer, Galligo and Stillman).