MA5 ALGEBRAIC GEOMETRY - HOMEWORK 5

DUE FRIDAY 19/3, 12PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

(1) Compute the Hilbert polynomials of the dth Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^d$ for $d = 1, 2, 3, 4$. What is the dimension of the image in each case? Conjecture a general formula for the Hilbert polynomial. Harder Bonus (Optional) Question: Can you prove this?

(2) Compute the dimension of $\mathbb{P}^1 \times \mathbb{P}^2$, by computing the Hilbert polynomial of the Segre embedding.

(3) (Optional) We gave in class a definition of $I_{d,n} = \langle p_{JK} : J, K \subseteq \{1, \ldots, n\}, |J| = d-1, |K| = d+1\rangle$, where $p_{JK} = \sum_{k \in K} (-1)^{\text{sign}(k)} x_{J \cup k} x_{K \setminus k}$. Show that $I_{2,4}$ is principal.

(4) Show that $G_{2,5}$ is smooth.

(5) Let X be a d-dimensional projective variety with Hilbert polynomial $P(t) = \sum_{i=0}^{d} a_i t^i$. The *degree* of X is $a_d d!$.
   (a) Show that the degree of the twisted cubic (the image of the 3rd Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$) is 3.
   (b) Show that the degree of $\mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding is two. Use your answer to Question 2 to compute the degree of $\mathbb{P}^1 \times \mathbb{P}^2$ in the Segre embedding.
   (c) If X is a d-dimensional variety, most $(n-d)$-dimensional subspaces in $\mathbb{P}^n$ intersect X in a finite number of points. Show this for X being the image of the dth Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^n$.
   (d) Show that for the dth Veronese embedding of $\mathbb{P}^1$ the number of such points is equal to d. (The number is equal to the degree in general - much harder, non-homework, challenge question: prove this).