MA5 ALGEBRAIC GEOMETRY - HOMEWORK 4

DUE FRIDAY 12/3, 12PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

1. Let $\phi : X \to Y$ and $\psi : Y \to Z$ be dominant morphisms, where $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, and $Z \subset \mathbb{A}^p$ are affine varieties. Show that the composition $\psi \circ \phi : X \to Z$ is a dominant morphism.
2. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, and $Z \subset \mathbb{A}^p$ be irreducible affine varieties, and let $\phi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow Z$ be dominant rational maps. Show that there is a rational map $X \dashrightarrow Z$ which equals $\psi \circ \phi$ on a nonempty open subset $U \subset X$.
3. Show that an ideal $I \subset \mathbb{k}[x_0, \dots, x_n]$ is homogeneous if and only if for all $f \in I$, each homogeneous piece of f lies in I.
4. Show that a single point in $\mathbb{P}^n$ is Zariski closed by giving its defining ideal.
5. Consider the rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ given by $\phi([x_0 : x_1 : x_2 : x_3]) = [x_0x_1 : x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3 : x_2x_3]$.
   (a) Does ϕ define a morphism $\mathbb{P}^3 \to \mathbb{P}^5$?
   (b) Consider $X = \mathbb{V}(x_0^2 + x_1^2 + x_2^2 + x_3^2) \subset \mathbb{P}^3$. Does ϕ induce a morphism $\phi : X \to \mathbb{P}^5$?
6. Describe the irreducible components of the projective variety $X = \mathbb{V}(x_0x_1 - x_2x_3, x_0x_2 - x_1x_3) \subset \mathbb{P}^3$.
7. Let α be a root of the polynomial $x^2 + a_1x + a_0 \in \mathbb{Q}[x]$, and β be a root of the polynomial $x^2 + b_1x + b_0 \in \mathbb{Q}[x]$ (so $a_i, b_j \in \mathbb{Q}$). Write down polynomials with $\alpha + \beta$ and $\alpha\beta$ as roots. (This question is to force those of you who just asked Mathematica for the minimal polynomial last week to use Gröbner bases...)
8. Find the projective closure of the following affine varieties when we identify $\mathbb{A}^n$ with U_0. For each variety, which points are added to take the closure?
   (a) $X = V(x_1^2 - x_2, x_1^3 - 2x_3^2) \subset \mathbb{A}^3$.
   (b) $X = \{(t, t^2, t^4) : t \in \mathbb{k}\} \subset \mathbb{A}^3$ (check that this is an affine variety!)
   (c) $X = V(-x_1 + 3x_2 - 8, -x_2^2 + 7x_2 - 12, x_1x_2 - 4x_1 - x_2 + 4)$

9. Compute the Hilbert polynomial of the ideal $I = \langle x_0^2 x_1, x_0 x_3, x_1 x_2^2, x_2^5 \rangle \subseteq \Bbbk[x_0, x_1, x_2, x_3]$. What is the dimension of $\mathbb{V}(I)$?
10. Let $X = \overline{\mathrm{im}\phi}$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ is given by $\phi([x_0 : x_1]) = [x_0^4 : x_0^3 x_1 : x_0 x_1^3 : x_1^4]$. Compute the Hilbert polynomial of X. What is the dimension of X?