MA5 ALGEBRAIC GEOMETRY - HOMEWORK 3

DUE FRIDAY 26/2, 12PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

(1) (a) Let $I = \langle f_1, \ldots, f_s \rangle \subset \Bbbk[x_1, \ldots, x_n]$. Show that $f \in \sqrt{I}$ if and only if $1 \in \langle f_1, \ldots, f_s, 1 - yf \rangle \subseteq \Bbbk[x_1, \ldots, x_n, y]$.
   (b) Let $I = \langle xy^2 + 2y^2, x^4 - 2x^2 + 1 \rangle \subset \Bbbk[x, y]$. Let $f = y - x^2 + 1$. Is $f \in \sqrt{I}$? What about $g = x^2 - 2$?

(2) Let I, J be ideals in $\Bbbk[x_1, \ldots, x_n]$.
   (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
   (b) Give an example to show that we do not always have $\sqrt{I + J} = \sqrt{I} + \sqrt{J}$.

(3) Let I, J, I_i, J_i, and K be ideals in $\Bbbk[x_1, \ldots, x_n]$ for $1 \leq i \leq r$. Prove the following identities for colon ideals:
   (a) $((\bigcap_{i=1}^r I_i) : J) = \bigcap_{i=1}^r (I_i : J)$,
   (b) $(I : (\sum_{i=1}^r J_i) = \bigcap_{i=1}^r (I : J_i)$,
   (c) $(I : J) : K = (I : JK)$.

(4) (General linear group) Identify $\mathbb{A}^{n^2}$ with the space of $n \times n$ matrices $A = (a_{ij})$ with coordinate ring $\Bbbk[a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}]$.
   (a) Show that matrix multiplication induces a morphism $\mu : \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \to \mathbb{A}^{n^2}$ given by $(A, B) \mapsto AB$.
   (b) Show that there is a rational map $i : \mathbb{A}^{n^2} \dashrightarrow \mathbb{A}^{n^2}$ given by $A \mapsto A^{-1}$.

(5) Is the intersection of two irreducible varieties irreducible? Give a proof or counterexample.

(6) Let $X = V(x^3 - y^2) \subset \mathbb{A}^2$ and $Y = V(u^3 - v^4) \subset \mathbb{A}^2$ be two varieties.
   (a) Show that $\Bbbk(X) \cong \Bbbk(Y)$.
   (b) Construct an explicit birational map $\phi : X \dashrightarrow Y$.

(7) Let $I = \langle x^{u_1}, \ldots, x^{u_s} \rangle$ and $J = \langle x^{v_1}, \ldots, x^{v_r} \rangle$ be monomial ideals.
   (a) Show that $I \cap J = \langle \mathrm{lcm}(x^{u_i}, x^{v_j}) : 1 \leq i \leq s, 1 \leq j \leq r \rangle$. Hint: a polynomial is in a monomial ideal if and only if each term lives in the ideal.

(b) Show that $(I : x_i) = \langle x^{u_j - \mathbf{e}_i} : (u_j)_i > 0\rangle + \langle x^{u_j} : (u_j)_i = 0\rangle$.

(c) Show that $I = \sqrt{I}$ if and only if every minimal generator x^u satisfies $u_i \in \{0, 1\}$.

(d) Show that I is prime if and only if it is generated by a collection of the variables.

(8) Give an algorithm to write a radical monomial ideal as the interesection of prime monomial ideals. Illustrate your algorithm on the ideal $I = \langle x_2x_4, x_2x_3, x_1x_2, x_1x_3x_4\rangle$.