MA5 ALGEBRAIC GEOMETRY - HOMEWORK 2

DUE FRIDAY 12/2, 12PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

(1) Let $\pi : \mathbb{A}^n \to \mathbb{A}^m$, for $m < n$, be projection onto the last m coordinates (so $\pi(x_1, \ldots, x_n) = (x_{n-m+1}, \ldots, x_n)$. Let $X \subset \mathbb{A}^n$ be a subvariety, and let $I = I(X) \subset \Bbbk[x_1, \ldots, x_n]$. (Note this is a slightly easier statement than the worksheet, so as not to assume the Nullstellensatz).
  (a) Let $J = I \cap \Bbbk[x_{n-m+1}, \ldots, x_n]$. Show that the closure of $\pi(X)$ in $\mathbb{A}^m$ is equal to $V(J)$.
  (b) Let $\prec$ be the lexicographic order on $\Bbbk[x_1, \ldots, x_n]$. Show that if $\mathrm{in}_{\prec}(f) \in \Bbbk[x_{n-m+1}, \ldots, x_n]$ then $f \in \Bbbk[x_{n-m+1}, \ldots, x_n]$.
  (c) Conclude that if $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis for I with respect to $\prec$, then $J = \langle g_i : g_i \in \Bbbk[x_{n-m+1}, \ldots, x_n] \rangle$.

(2) Let $\phi : \mathbb{A}^1 \to \mathbb{A}^4$ be the morphism given by $\phi(t) = (t^2, t^3, t^5, t^6)$. Find equations for $\overline{\mathrm{im}\phi}$. Did we need to take the closure here?

(3) Let $f = x^5 + 3x^4 - 2x^3 - 3x^2 - ax - 5$, and $g = x^5 + 7x4 - 5x^3 + ax^2 + x - 8$. For how many different values of a do f and g have a common factor?

(4) Let $f = ax^3 + bx^2 + cx + d$. Compute the resultant $\mathrm{Res}(f, f', x)$. Compare this with the classical formula for the discriminant of a cubic (see eg Wikipedia).

(5) Let $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{i=0}^{l} b_i x^i$ be two polynomials in $\Bbbk[x]$ with $\Bbbk$ algebraically closed. Let $\alpha_1, \ldots, \alpha_m$ be the roots of f, and let $\beta_1, \ldots, \beta_l$ be the roots of g.
  (a) Show that the a_i, b_j can be written as functions of α_i, β_j and a_m, b_l.
  (b) Let ψ denote the induced $\Bbbk$-algebra homomorphism $\Bbbk[a_0, \ldots, a_m, b_0, \ldots, b_l] \to \Bbbk[a_m, b_l, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l]$. Let $R = \psi(\mathrm{Res}(f, g, x))$. Compute R explicitly when f and g both have degree two. Factor your answer.
  (c) Show that for general $f, g \in \Bbbk[x]$ we have $S = a_m^l b_l^m \prod_{i=1,\ldots,m, j=1,\ldots,l} (\alpha_i - \beta_j)$ dividing R.

(d) Conclude that S and R agree up to a constant factor. Hint: Consider the degrees of the polynomials in subsets of the variables.

(6) Let $f, g \in \mathbb{k}[x_1, \dots, x_n]$, and suppose that there exist $A, B \in \mathbb{k}(x_2, \dots, x_n)[x_1]$ with $Af + Bg = 1$. Show that we can choose such A, B to have denominator dividing $\mathrm{Res}(f, g, x_1)$, so $A\mathrm{Res}(f, g, x_1), B\mathrm{Res}(f, g, x_1) \in \mathbb{k}[x_2, \dots, x_n]$. Conclude that there are $A, B \in \mathbb{k}[x_2, \dots, x_n]$ with $Af + Bg = \mathrm{Res}(f, g, x_1)$. Hint: Cramer's rule.

(7) Let $\alpha = \sqrt[3]{(3)} + \sqrt{(7)}\sqrt[4]{(2)}$. Compute a polynomial $f \in \mathbb{Q}[x]$ of minimal degree with $f(\alpha) = 0$. Show/explain your work. Hint: Use elimination theory and a computer!

(8) Let $\phi^* : \mathbb{k}[x_1, \dots, x_n] \to \mathbb{k}[z_1, \dots, z_n]$ be given by $\phi^*(x_i) = z_i + a_i z_1$ for fixed $a_i \in \mathbb{k}$.

(a) Show that ϕ^* is an isomorphism of rings.

(b) Let ϕ be the induced map $\mathbb{A}^n \to \mathbb{A}^n$. When $n = 3$, what is $\phi((1, 2, 3))$?

(c) Let $X \subset \mathbb{A}^n$ be a subvariety. Show that $\phi : X \to \phi(X)$ is an isomorphism.

(d) Conclude in particular that if $X = \emptyset$, then $\phi(X) = \emptyset$.