MA5 ALGEBRAIC GEOMETRY - HOMEWORK 1

DUE FRIDAY 29/1, 12PM

Questions 2 and 4 are optional.

(1) Let $f = x^2y + xy^2 + y^2$, and $\prec$ be the lexicographic order with $x \succ y$. Apply the division algorithm to find the remainder on dividing f by $\{xy - 1, y^2 - 1\}$. Repeat with the polynomials in the order $\{y^2 - 1, xy - 1\}$. Do these two polynomials for a Gröbner basis for the ideal they generate?

(2) The algorithm to compute Gröbner bases is called Buchberger's algorithm. It can be thought of as a generalization of Gaussian elimination. The key step is the formation of the S-polynomial of two polynomials $f, g \in I$:

$$S(f,g) = \operatorname{lcm}(\operatorname{in}_\prec(f), \operatorname{in}_\prec(g))/\operatorname{in}_\prec(f) f - \operatorname{lcm}(\operatorname{in}_\prec(f), \operatorname{in}_\prec(g))/\operatorname{in}_\prec(g) g.$$

The Buchberger algorithm then proceeds by computing the S-polynomial of any pair of generators for I, finding the remainder on dividing this polynomial by the generators, and adding this remainder to the generating set if nonzero. This procedure continues until the generating set stabilizes, at which point the generating set is a Gröbner basis.

(a) Read a proof that this algorithm works (for example, in Cox, Little, O'Shea, or Hassett).

(b) Use it to compute a Gröbner basis for the ideal $I = \langle x^2, y^2, xy + yz \rangle \subset k[x, y, z]$ with respect to the revlex term order. (Eisenbud *Commutative Algebra*, Ex 15.27).

(3) Fix a term order $\prec$ on the polynomial ring $K[x_1, \dots, x_n]$. Recall that $\mathcal{G} = \{g_1, \dots, g_s\}$ is a *reduced* Gröbner basis for an ideal I if $\mathcal{G}$ is a Gröbner basis, $\{\operatorname{in}_\prec(g_1), \dots, \operatorname{in}_\prec(g_s)\}$ is an irredundant (no repeats) minimal generating set for $\operatorname{in}_\prec(I)$, and for each g_i, no term of g_i other than its initial term is divisible by $\operatorname{in}_\prec(g_j)$ for any $1 \leq j \leq s$. Show that every ideal has a unique reduced Gröbner basis.

(4) An ideal $I \subset \Bbbk[x_1, \dots, x_n]$ is *radical* if $I = \{f \in \Bbbk[x_1, \dots, x_n] : f^k \in I \text{ for some } k > 0\}$.

(a) Let $I \subset \Bbbk[x_1, \dots, x_n]$ be an ideal. Show that if $\operatorname{in}_\prec(I)$ is radical, then I is radical.

(b) Given an example to show that the converse does not hold; that is, give an example of a radical ideal all of whose initial ideals are radical.
(c) Compute the radical of the ideal $\langle x_1^2, x_1x_2^3, x_2^2x_3\rangle \subset \Bbbk[x_1, x_2, x_3]$.
(d) Characterize which monomial ideals are radical.

(5) (Hassett #3.7) Recall that the closure $\overline{S}$ of a set S in a topological space is the smallest closed set containing S. Compute the Zariski closures $\overline{S} \subset \mathbb{A}^2$ for $K = \mathbb{Q}$ for the following sets:
(a) $S = \{(n^2, n^3) : n \in \mathbb{N}\}$
(b) $S = \{(x, y) : x^2 + y^2 < 1\}$
(c) $S = \{(x, y) : x + y \in \mathbb{Z}\}$

(6) Let $R = \mathbb{C}[x_{ij} : 1 \leq i \leq 4, 1 \leq j \leq 4]$. Let $V = \{X = (x_{ij}) : X \text{ is a } 4 \times 4 \text{ matrix of rank } 1\}$. Show that V is a subvariety of $\mathbb{A}^{16}$.

Recall (or convince yourself if you don't already know this) that every 4×4 matrix of rank at most one has the form $X = \mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{C}^4$. Use this to describe a map $\phi : \mathbb{A}^8 \to \mathbb{A}^{16}$ whose image is X.

(7) Learn to use a computer algebra system that computes Gröbner bases. I highly recommend `Macaulay 2`, which is freely available at: http://www.math.uiuc.edu/Macaulay2/. An older version packaged for windows is at: http://www.commalg.org/m2win/. Another option is `Sage`, freely available from http://www.sagemath.org/. In either case there are tutorials available from the main webpage.

Attach a printout of the Gröbner bases for $I = \langle x^5 + y^4 + z^3, x^3 + y^2 + z^2 - 1\rangle$ with respect to the revlex and lexicographic orders.

(8) (CLO p99 #11) Let $X = V(x + y + z - 3, x^2 + y^2 + z^2 - 5, x^3 + y^3 + z^3 - 7)$.
(a) Show that $x^4 + y^4 + z^4 = 9$ for all $(x, y, z) \in X$.
(b) What is the value of $x^5 + y^5 + z^5$ for $(x, y, z) \in X$? (Hint: Compute remainders)
(c) (Harder) What is the size of X? (you may answer for either the field $K = \mathbb{R}$, or $K = \mathbb{C}$). (Hint: Compute a lexicographic Gröbner basis - you will probably want to use a computer).