MA5 ALGEBRAIC GEOMETRY - HOMEWORK 6

NOT ASSESSED

These problems vary widely in difficulty. The first 6 are more routine, and you should all be able to do them. Question 7 is less routine, but also doable. The other questions are harder; you will learn something from attempting them, and the definitions are examinable.

This homework covers material that has not been assessed yet, so some fraction of this material is guaranteed to be on the exam.

Let me know as soon as you find something that you think might be a mistake (even if you think the correction is "obvious").

(1) Let $I = \langle x_0^3, x_1x_2, x_0x_2, x_1^2 \rangle \subset K[x_0, x_1, x_2]$. Show that $(I : x_0) = \langle x_0^2, x_1^2, x_2 \rangle$.

(2) Describe how to compute $(I : x_i)$ when I is a monomial ideal (ie state a proposition and prove it).

(3) Let $I = \langle x_0^2, x_1^2x_2, x_2^2x_3 \rangle \subseteq S := K[x_0, x_1, x_2, x_3]$. Compute the Hilbert polynomial of S/I.

(4) Let $I = \langle x_0x_1 - x_2^2, x_1^3 - x_3^3 \rangle \subseteq S := K[x_0, x_1, x_2, x_3]$. Compute the Hilbert polynomial of S/I.

(5) Show that the intersection of two radical ideals is radical. Use this to show that if $X = \{p_1, \ldots, p_s\}$ is a collection of points in $\mathbb{P}^n$, then $I_X = \cap_{i=1}^s I_{p_i}$ (assuming K is algebraically closed).

(6) Let $p_1 = [1 : 2 : 3]$, $p_2 = [1 : 1 : 0]$, and $p_3 = [0 : 1 : 2]$. Let $X = \{p_1, p_2, p_3\}$. Compute I_X (you may want to use a computer). Compute the Hilbert polynomial P_X directly from I_X. (Hint: you already know the answer, so can check your work).

(7) Let $X \subset \mathbb{A}^n$ be an irreducible variety. Show that if $\dim(X) = 0$ then X is a point. Hint: If $\dim(X) = 0$ then $K(X)$ is algebraic over K. Conclude that if X is an arbitrary affine or projective variety with $\dim(X) = 0$ then X is a finite collection of points.

(8) In this question you will show that the dimension of a projective variety equals the degree of the Hilbert polynomial.

   (a) Let $X \subset \mathbb{A}^n$ be a variety, and let $H = V(\ell)$, where $\ell = a_0 + \sum a_i x_i$ for some $a_0, \ldots, a_n \in K$. Show that if no irreducible component of X is contained in H, then $\dim(X \cap H) \leq$

$\dim(X) - 1$. Why can you always choose an H satisfying this condition?

(b) Let $X \subset \mathbb{P}^n$ be a projective variety, and $H = \mathbb{V}(\ell)$ with $\ell = \sum_{i=0}^{n} a_i x_i$ for some $a_0, \dots, a_n \in K$. Show that there is an open set $U \subset \mathbb{P}^n$ for which if $[a_0 : a_1 : \cdots : a_n] \in U$ then $P_{X \cap H}(t) = P_X(t) - P_X(t-1)$, so $\deg(P_{X \cap H}) = \deg(P_X) - 1$. Hint: this requires some of the commutative algebra of nonzerodivisors.

(c) Combine the previous parts to conclude that if X is a subvariety of $\mathbb{P}^n$, then $\dim(X) \geq \deg(P_X)$, where P_X is the Hilbert Polynomial of X.

(d) Use the fact that if $X \subset \mathbb{A}^n$ with $\dim(X) = d$ then there is $\sigma = \{i_1, \dots, i_d\}$ with $i_X \cap K[x_j : j \in \sigma] = 0$ to show that $\deg(P_X) \geq \dim(X)$. Hint: consider a lexicographic initial ideal with $\{x_j : j \in \sigma\}$ last in the order.

(e) Conclude that $\dim(X) = \deg(P_X)$.

(f) Conclude that we actually have equality for general H in part 8a. This is a special case of Krull's principal ideal theorem; see, for example, chapter 10 of Eisenbud's *Commutative Algebra*.)

(9) Let X be a d-dimensional projective variety with Hilbert polynomial $P(t) = \sum_{i=0}^{d} a_i t^i$. The *degree* of X is $a_d d!$.

(a) Show that the degree of the twisted cubic (the image of the 3rd Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$) is 3. (Hint: we computed the Hilbert polynomial in class).

(b) Show that the degree of $\mathbb{P}^1 \times \mathbb{P}^1$ in the Segre embedding is two.

(c) If X is an r-dimensional variety, a generic $(n-r)$-dimensional subspaces in $\mathbb{P}^n$ intersect X in a finite number of points. show this for X being the image of the dth Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^n$.

(d) Show that for the dth Veronese embedding of $\mathbb{P}^1$ the number of such points is equal to d.

(10) Use the approach of Question 8 (intersecting with a general hyperplane) to show that the degree of a d-dimensional irreducible projective variety X equals the number of intersection points of X with a generic subspace of dimension $n - d$ (ie there is an open set $U \subset G(n - d + 1, n + 1)$ with $L \in U$ implying that $|X \cap L|$ is the degree of X.

(11) Show that if X_1 is an irreducible component of X, and X_1 is the unique component of X containing a point $\mathbf{a} \in X$, then $T_{\mathbf{a}}(X) = T_{\mathbf{a}}(X_1)$.

(12) Let X be an irreducible variety in $\mathbb{A}^n$. The goal of this exercise is to show that for all $\mathbf{a} \in X$ we have $\dim(T_{\mathbf{a}}(X)) \geq \dim(X)$. Recall that $T_{\mathbf{a}}(X)$ is an affine subspace (ie of the form $\mathbf{a} + L$ for a subspace L), so the dimension on the left is in the sense of linear algebra, while the dimension on the right is in the sense introduced in this module. The proof is by induction on $\dim(X)$.
  (a) Show that the base case follows from Question 7.
  (b) Show that there is an open subset $U \subset \mathbb{P}^n$ for which if $\mathbf{b} \in U$ then $H = V(\sum_{i=0}^n b_i x_i)$ satisfies $\operatorname{rank}(\operatorname{Jac}(X)|_{x=\mathbf{a}}) = \operatorname{rank}(\operatorname{Jac}(X \cap H)|_{x=\mathbf{a}}) - 1$ for all $\mathbf{a} \in X \cap H$.
  (c) Use Question 8 to conclude that $\dim(T_{\mathbf{a}}(X)) \geq \dim(X)$.

(13) Let $X \subseteq \mathbb{A}^n$ be an affine variety. Use Question 12 to show that the set $\{a \in \mathbb{A}^n : X \text{ is singular at } a\}$ is a Zariski closed set. This is called the *singular locus* of X.

(14) Let $X = \{p_1, \dots, p_s\}$ be a finite collection of points in $\mathbb{P}^n$. Show that the Hilbert function $H_X(d)$ is the dimension of the vector space of functions $\{p_1, \dots, p_s\} \to K$ that are restrictions of polynomials of degree d in $K[x_0, \dots, x_n]$.

(15) The goal of this question is to apply some basic properties of Hilbert polynomials to prove some classical theorems from algebraic and projective geometry. The questions are in reverse chronological order (ie newest first), though the statements are the modern formulations. For hints, and many nice extensions, see *Cayley-Bacharach theorems and conjectures* by Eisenbud, Green, Harris, Bulletin of the AMS, 33, 1996. (link).
  (a) (Chasles 1885). Let $f, g \in \mathbb{C}[x_0, x_1, x_2]$ be two polynomials of degree three with $\mathbb{V}(f) \cap \mathbb{V}(g)$ equal to nine distinct points (recall that by Bézout we knew that as long as $\gcd(f, g) = 1$ then this number was at most nine). Show that if h is any other homogeneous polynomial of degree three that vanishes at eight of these points, then it must also vanish at the ninth. Hint: Use the previous question. If Y is eight of the nine points, and X is all of them, why do you know that $H_X(3) = H_Y(3)$?
  (b) (Pascal 1640). A conic in $\mathbb{P}^2$ is a variety of the form $\mathbb{V}(f)$ for $f \in K[x_0, x_1, x_2]$ of degree two (for example, a circle). Show that if a hexagon is inscribed in a conic in $\mathbb{P}^2$, then the opposite sides of the hexagon meet in three collinear

points. This hexagon need not be convex, so the question really asks: if $p_1, q_2, p_3, q_1, p_2, q_3$ are six distinct points on the conic, and r_i is the intersection point of $\overline{p_i q_j}$ and $\overline{p_j q_i}$ for $\{1,2,3\} = \{i,j,k\}$, then r_1, r_2, r_3 are collinear (the strange ordering of the vertices of the hexagon is chosen to make this description simpler). Hint: Show that this is a corollary of Chasles theorem. Your cubic polynomials do not need to be irreducible; they could be a product of a polynomial of degree two (a "quadric") and one of degree one, or of three polynomials of degree one.

(c) (Pappus, 4th Century AD). Let L and M be two lines in the plane. Let p_1, p_2, p_3 be distinct points of L, and let q_1, q_2, q_3 be distinct points of M, all distinct from the point $L \cap M$. For $1 \leq j < k \leq 3$ let r_{jk} be the point of intersection of the lines $\overline{p_j q_k}$ and $\overline{p_k q_j}$. Show that r_{12}, r_{13}, and r_{23} are collinear. Hint: Deduce this from Pascal's theorem. Your proof should of Pascal's theorem should not have needed the conic $\mathbb{V}(f)$ to be irreducible.