MA5 ALGEBRAIC GEOMETRY - HOMEWORK 4

DUE TUESDAY 25TH NOVEMBER, 2PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

A: Warm-up problems

(1) Show that $V(I(V(I)) = V(I)$, for any ideal I, and that $I(V(I(X))) = I(X)$ for any set X. This is true over an arbitrary field.
(2) Show that a prime ideal is radical.
(3) Show that if $X_1, \dots, X_r$ are varieties with $X_i \subsetneq X$ and $X = \cup_{i=1}^r X_i$ then X is reducible (ie we can find proper subvarieties Y_1, Y_2 with $Y_1 \cup Y_2 = X$).
(4) Show that an ideal $I \subset \Bbbk[x_0, \dots, x_n]$ is homogeneous if and only if for all $f \in I$, each homogeneous piece of f lies in I.
(5) Let $I = \langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2 \rangle$, and let $X = \mathbb{V}(I) \subseteq \mathbb{P}^3$. Let $X_i = X \cap U_i$ for $i = 0, 1, 2, 3$. Identify U_i with $\mathbb{A}^3$, so we can regard each X_i as a subvariety of $\mathbb{A}^3$. Show explicity that the X_i are all birational by constructing explicit isomorphisms of their field of fractions.
(6) Fix $a, b, \in \Bbbk$, and let $L = V(y - ax - b)$ be a line in $\mathbb{A}^2$. What point(s) are added in the projective closure $\overline{L}$ of L?

B: Exercises

(1) Let $\phi : X \to Y$ and $\psi : Y \to Z$ be dominant morphisms, where $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, and $Z \subset \mathbb{A}^p$ are affine varieties. Show that the composition $\psi \circ \phi : X \to Z$ is a dominant morphism.
(2) Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, and $Z \subset \mathbb{A}^p$ be irreducible affine varieties, and let $\phi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow Z$ be dominant rational maps. Show that there is a rational map $X \dashrightarrow Z$ which equals $\psi \circ \phi$ on a nonempty open subset $U \subset X$.
(3) Let $X \subset \mathbb{A}^n$ be a variety, and $\phi \colon X \dashrightarrow \mathbb{A}^m$ be a rational map. Show that the image imϕ is well-defined, by showing that if $U \subset \mathbb{A}^n$ and $U' \subset \mathbb{A}^n$ are open sets on which ϕ is defined, then $\overline{\phi(U \cap X)} = \overline{\phi(U' \cap X)}$.

(4) Consider the rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ given by $\phi([x_0 : x_1 : x_2 : x_3]) = [x_0x_1 : x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3 : x_2x_3]$.
   (a) Does ϕ define a morphism $\mathbb{P}^3 \to \mathbb{P}^5$?
   (b) Consider $X = \mathbb{V}(x_0^2 + x_1^2 + x_2^2 + x_3^2) \subset \mathbb{P}^3$. Does ϕ induce a morphism $\phi : X \to \mathbb{P}^5$?

(5) Find the projective closure of the following affine varieties when we identify $\mathbb{A}^n$ with U_0. For each variety, which points are added to take the closure?
   (a) $X = V(x_1^2 - x_2, x_1^3 - 2x_3^2) \subset \mathbb{A}^3$.
   (b) $X = \{(t, t^2, t^4) : t \in \Bbbk\} \subset \mathbb{A}^3$ (check that this is an affine variety!)
   (c) $X = V(-x_1 + 3x_2 - 8, -x_2^2 + 7x_2 - 12, x_1x_2 - 4x_1 - x_2 + 4)$

(6) Describe the irreducible components of the projective variety $X = \mathbb{V}(x_0x_1 - x_2x_3, x_0x_2 - x_1x_3) \subset \mathbb{P}^3$.

(7) Is the intersection of two irreducible varieties irreducible? Give a proof or counterexample.

C: Extensions

(1) Show that there is a unique element of $\mathrm{Aut}(\mathbb{P}^1)$ that takes any three distinct points in $\mathbb{P}^1$ to any other three points in $\mathbb{P}^1$. For example, given $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \in \mathbb{P}^1$, there is a unique $\phi \in \mathrm{Aut}(\mathbb{P}^1)$ with $\phi(\mathbf{x}) = [1 : 0]$, $\phi(\mathbf{y}) = [0 : 1]$, and $\phi(\mathbf{z}) = [1 : 1]$.

(2) Let $f \in \mathbb{C}[x]$ be a polynomial of degree three, and let $\tilde{f} \in \mathbb{C}[x, y]$ be its homogeneization. Use the previous part to show that if f has three distinct roots then there is a linear change of coordinates $x' = ax + by$, $y' = cx + dy$ for some a, b, c, d with $ad - bc \neq 0$ which takes $\tilde{f}$ to $x(x - y)(x + y)$.

(3) (Challenge:) By considering explicitly the form of a, b, c, d, show that the roots of f have solutions that can be expressed in radicals of its coefficients.