MA5 ALGEBRAIC GEOMETRY - HOMEWORK 3

DUE THURSDAY 6TH NOVEMBER, 2PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

A: Warm-up problems

(1) Show that $V(I(V(I)) = V(I)$, for any ideal I, and that $I(V(I(X))) = I(X)$ for any set X. This is true over an arbitrary field.
(2) Show that a single point in $\mathbb{A}^n$ is Zariski closed by giving its defining ideal (ie an ideal I such that $V(I)$ is the point p).
(3) Let $X = V(yz - 1, x^2 + y^2 + z^2) \subseteq \mathbb{A}^3$, and let $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ be given by $\pi(x, y, z) = (y, z)$. Calculate $\overline{\pi(X)}$. Show that $\overline{\pi(X)} = \pi(X)$.
(4) Let $X = V(xy - x + 2z, 3x^2y - z^3) \subseteq \mathbb{A}^3$, and let $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ be given by $\pi(x, y, z) = (y, z)$. Calculate $\overline{\pi(X)}$. Show that $\overline{\pi(X)} = \pi(X)$.
(5) Check that $1 \in \sqrt{I}$ if and only if $I = \langle 1 \rangle$.
(6) Let $I = \langle x^2 - 2x + 1, y^3 - 3y^2 + 3y - 1 \rangle \subset \Bbbk[x, y]$. Calculate $\sqrt{I}$.
(7) We saw that the Nullstellensatz fails when, for example, $\Bbbk = \mathbb{R}$ and $I = \langle x^2 + 1 \rangle \subseteq \mathbb{R}[x]$. Give an example to show that the Nullstellensatz can fail when $\Bbbk$ is not algebraically closed even when the variety $X \subseteq \mathbb{A}^n$ is not empty.

B: Exercises

(1) (Hassett 4.1) The cardioid is defined as the curve $C \subset \mathbb{R}^2$ with parametric representation

$$x(\theta) = \cos(\theta) + 1/2 \cos(2\theta),$$

$$y(\theta) = \sin(\theta) + 1/2 \sin(2\theta),$$

for $0 \leq \theta \leq 2\pi$. Show that C can be defined by a polynomial equation $p(x, y) = 0$. Hint: Introduce auxiliary variables u and v satisfying $u^2+v^2 = 1$. Express x and y as polynomials in u and v, and eliminate u and v to get the desired equation in x and y.

You will probably want to use a computer to finish this calculation. In Macaulay 2 the key command is `eliminate(u,v,I)`, where I is the relevant ideal.

(2) Let $X = V(x^3 - y^2) \subset \mathbb{A}^2$ and $Y = V(u^3 - v^4) \subset \mathbb{A}^2$ be two varieties.
  (a) Show that $\Bbbk(X) \cong \Bbbk(Y)$.
  (b) Construct an explicit birational map $\phi : X \dashrightarrow Y$.

(3) (a) Let $I = \langle f_1, \ldots, f_s \rangle \subset \Bbbk[x_1, \ldots, x_n]$. Show that $f \in \sqrt{I}$ if and only if $1 \in \langle f_1, \ldots, f_s, 1 - yf \rangle \subseteq \Bbbk[x_1, \ldots, x_n, y]$.
  (b) Let $I = \langle xy^2 + 2y^2, x^4 - 2x^2 + 1 \rangle \subset \Bbbk[x, y]$. Let $f = y - x^2 + 1$. Is $f \in \sqrt{I}$? What about $g = x^2 - 2$?

(4) Let I, J be ideals in $\Bbbk[x_1, \ldots, x_n]$.
  (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
  (b) Give an example to show that we do not always have $\sqrt{I + J} = \sqrt{I} + \sqrt{J}$.

(5) (General linear group) Identify $\mathbb{A}^{n^2}$ with the space of $n \times n$ matrices $A = (a_{ij})$ with coordinate ring $\Bbbk[a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}]$.
  (a) Show that matrix multiplication induces a morphism $\mu : \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \to \mathbb{A}^{n^2}$ given by $(A, B) \mapsto AB$.
  (b) Show that there is a rational map $i : \mathbb{A}^{n^2} \dashrightarrow \mathbb{A}^{n^2}$ given by $A \mapsto A^{-1}$.

C: Extensions

(1) Let I be an ideal in $K[x]$ where K is an arbitrary field (ie not necessarily algebraically closed). Give an algorithm to compute the radical $\sqrt{I}$. Hint: One can take a (formal) derivative f' of a polynomial f with coefficients in any field. Think about the gcd of f and f'. Your algorithm will show that if, for example, $K = \mathbb{Q}$, the radical of $\langle f \rangle$ is generated by a polynomial with coefficients in $\mathbb{Q}$ even if f has no rational roots.

(2) (Important) Explain how to compute the closure of the image of a rational map (ie generalize the construction we did for morphisms). Check your description by computing the image of the rational map from $\mathbb{A}^1$ to $\mathbb{A}^2$ whose image is the circle $V(x^2 + y^2 - 1)$ discussed in class.