FACTS ABOUT SHORTEST PATHS

DIANE MACLAGAN

This handout contains some facts about shortest paths in preparation for the Bellman-Ford algorithm. We assume that all digraphs G are simple and there is a directed path from v to all other vertices.

Key facts about shortest paths

(1) If G is a weighted digraph with no negative cost circuits then there is a shortest path from v to any $u \in V$.
(2) In that case the shortest path from v to u is a *simple* path.

Proof. Fix a vertex $u \in V$. If there were no shortest path from v to u, then for all $W \in \mathbb{R}$ there would be a path of weight less than W from v to u. There are only finitely many simple paths from v to u. Let W_s be the smallest weight of a simple path from v to u. Choose $W < W_s$. Let $v = v_0, v_1, \dots, v_s = u$ be a path from v to u of weight $W' = \sum_{k=0}^{s-1} w((v_k, v_{k+1}))$ less than W. We may assume the path has been chosen to have as few vertices as possible out of all paths of weight less than W. Since $W < W_s$ this path is not simple, so there is $i < j$ with $v_i = v_j$. Then $v = v_0, v_1, \dots, v_i, v_{j+1}, \dots, v_s = u$ is a path from v to u with fewer vertices, so has weight $W'' > W'$. But then $W' - W'' = \sum_{k=i}^{j-1} w((v_k, v_{k+1})) < 0$, so $v_i, v_{i+1}, \dots, v_j = v_i$ is a negative cost circuit.

If there is no negative cost circuit and $v = v_0, \dots, v_s = u$ is a path from v to u with $v_i = v_j$ for some $i < j$, then $v = v_0, \dots, v_i, v_{j+1}, \dots, u$ is a path from v to u with at worst the same weight. Thus the shortest path from v to u is simple. □

Key facts about the Ford algorithm:

(1) If G has no negative cost circuit then at each stage, if $y_u \neq \infty$ then y_u is the cost of a simple path from v to u.
(2) If G has a negative cost circuit, then there is always an incorrect edge of the algorithm at each stage.

Proof. Suppose first that G has no negative cost circuit. Let y_u^j be the value of y_u after j steps of the algorithm (so j edges have been corrected). We need to show that if $y_u^j < \infty$ then y_u^j is the length of a simple path from v to u. We construct this path backwards as follows. Suppose y_u^j was last updated at step $j_0 \leq j$ by correcting the edge (v_1, u), so $y_u^j = y_{v_1}^{j_0} + w((v_1, u))$. Let $j_1 < j_0$ be the step at which $y_{v_1}^{j_0}$ obtained this value, by correcting the edge (v_2, v_1), so $y_{v_1} = y_{v_2} + w((v_2, v_1))$. Iterate. Since j_k decreases at each stage we must eventually have $j_s = 0$. The first edge corrected is always an edge from v, as all other $y_u^s = \infty$, so it is the only incorrect edge. Thus $v_s = v$, and $v = v_s, v_{s-1}, \ldots, v_1, u$ is a path from v to u.

We now show that if G has no negative cost circuits this is a simple path. Otherwise there is some loop with $v_i = v_j$ with $i < j$. Note that $y_{v_k}^{j_k} - y_{v_{k+1}}^{j_k} = w((v_{k+1}, v_k))$. So

$$\begin{aligned}\sum_{k=i}^{j-1} w((v_{k+1}, v_k)) &= \sum_{k=i}^{j-1} y_{v_k}^{j_k} - y_{v_{k+1}}^{j_k} \\ &= \sum_{k=i}^{j-1} y_{v_k}^{j_k} - y_{v_{k+1}}^{j_{k+1}} \\ &= y_{v_i}^{j_i} - y_{v_j}^{j_j} \\ &< 0\end{aligned}$$

since $i < j$ means that $j_i > j_j$, so $y_{v_i}^{j_i} < y_{v_i}^{j_j}$, as the y_v^j decreases each time

Thus G has a negative cost circuit.

Now suppose that G has a negative cost circuit $v_1, v_2, \ldots, v_s = v_1$ of weight $W < 0$. We claim that once one of the y_{v_i} is not ∞ then at least one edge (v_i, v_{i+1}) is incorrect at each stage. Otherwise we have $y_{v_{i+1}} \leq y_{v_i} + w((v_i, v_{i+1}))$ for all i, so $\sum_{i=1}^{s-1} y_{v_{i+1}} \leq \sum_{i=1}^{s-1}(y_{v_i} + w((v_i, v_{i+1}))) = \sum_{i=1}^{s-1} y_{v_{i+1}} + W$, which is a contradiction.

□

Comprehension exercises: Show that if G has no negative cost circuit then no edge (u, v) will ever be incorrect. If G has a negative cost circuit, does it mean that some edge (u, v) must be incorrect at each stage?