MA 243 HOMEWORK 9

DUE: THURSDAY, DECEMBER 6 2007, BY 12PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

1. Find an affine transformation of $\mathbb{A}^2$ taking the set $\{(0,0),(1,0),(1,1)\}$ to the set $\{(1,2),(1,3),(2,2)\}$.
2. Find a projective transformation of $\mathbb{P}^1$ taking the set $\{(1:0),(0:1),(1:1)\}$ to $\{(1:2),(1:3),(1:4)\}$. Repeat for the sets $\{(1:0),(1:2),(1:3)\}$ and $\{(0:1),(1:1),(1:5)\}$.
3. Show that an affine subspace of $\mathbb{A}^n$ dimension d is isomorphic to $\mathbb{A}^d$ (ie there is a map taking points to points, and lines to lines). Repeat this for projective subspaces. In particular, lines in $\mathbb{A}^n$ look like $\mathbb{A}^1$, and lines in $\mathbb{P}^n$ look like $\mathbb{P}^1$.

B: Exercises

1. Find the intersection of the following pairs of lines in $\mathbb{P}^2$: $L = W/\sim$, $L' = W'/\sim$, where
   (a) $W = \{\mathbf{x} \in \mathbb{R}^3 : x_0 + x_1 = 0\}, W' = \{\mathbf{x} \in \mathbb{R}^3 : 2x_0 + x_1 - x_2 = 0\}$
   (b) $W = \{\mathbf{x} \in \mathbb{R}^3 : x_2 = 0\}, W' = \{\mathbf{x} \in \mathbb{R}^3 : x_1 = x_2\}$
2. Find an affine transformation of $\mathbb{A}^2$ taking the set $\{(3,4),(4,6),(6,11)\}$ to $\{(1,-1),(2,1),(3,5)\}$.
3. Find a projective transformation of $\mathbb{P}^2$ taking the (ordered) list $\{(1:1:0),(1:0:1),(1:1:1),(0:1:1)\}$ of points to the (ordered) list $\{(1:0:0),(0:1:0),(0:0:1),(1:1:1)\}$.
4. Compute the cross-ratio $\{P,Q;R,S\}$ of the set $\{P=(1:0),Q=(1:1),R=(2:1),S=(1:2)\}$ of points in $\mathbb{P}^1$.
5. Recall that we embed $\mathbb{A}^n$ into $\mathbb{P}^n$ by sending $\mathbf{x}$ to $(1:\mathbf{x})$. Given an affine transformation $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, write down the corresponding projective transformation it extends to (this was given in class briefly). Let $S(\mathbf{x}) = A'\mathbf{x} + \mathbf{b}'$. Write down the composition $S \circ T$, and compare it with the result of composing the corresponding projective transformations.

6. Prove that 3 lines L, M, N of $\mathbb{P}^n$ that intersect in pairs are either concurrent (have a common point) or coplanar.

C: Extensions

1. We can define affine and projective space over any field k. Affine space $\mathbb{A}^n_k$ is the vector space k^n with affine transformations $T(x) = Ax+b$ where A is an $n \times n$ invertible matrix with entries in k, and $b \in k^n$. Projective space $\mathbb{P}^n$ is $(k^{n+1} \setminus \mathbf{0})/\sim$, where $\sim$ is defined as before: $\mathbf{v} \sim \lambda\mathbf{v}$ for all $\lambda \in k \setminus 0$.
   (a) Consider the case $k = \mathbb{F}_2$, the finite field with two elements. What do lines look like in $\mathbb{A}^2$?
   (b) What about $\mathbb{A}^3$?
   (c) Repeat this for $k = \mathbb{F}_3$, the finite field with three elements.
   (d) Look at the game described at `http://www.setgame.com/set/index.html`. Can you see a connection?
   (e) Let $k = \mathbb{F}_2$. List the points in $\mathbb{P}^2_k$. Draw a picture of all the lines in $\mathbb{P}^2_k$.