MA 243 HOMEWORK 3

DUE: THURSDAY, OCTOBER 25, 2007, BY 12PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

1. If $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, and $S(\mathbf{x}) = C\mathbf{x} + \mathbf{d}$, what is the composition $T \circ S$? What about $S \circ T$? Are they always the same (Hint: where is $\mathbf{0}$ taken in each case?)
2. Describe the following motions of $\mathbb{E}^2$ geometrically.
   (a) Reflection in the line $y = 0$ followed by the reflection in the line $y = x$.
   (b) Reflection in the line $y = 0$ followed by the reflection in the line $y = 1$.
   (c) Rotation by $\pi/2$ about the origin followed by translation by $(1, -1)$.
   (d) Rotation by $\pi/2$ about the origin followed by rotation by $\pi/2$ about the point $(2, 0)$.
   (e) Rotation by $\pi/2$ about the origin followed by reflection in the line $y = 0$.

B: Exercises

In class we showed that every motion of $\mathbb{E}^2$ is either a rotation, reflection, translation or glide reflection. In particular this means that any composition of two of these is one of these. You will check some cases explicitly in this exercise.

1. What is the composition of two reflections? Let T and S be two reflections about lines L and M.
   (a) Show that if L and M intersect then the composition $T \circ S$ is a rotation. What is the angle? Prove your answer.
   (b) Show that if L and M do not intersect then $T \circ S$ is translation. By what vector does it translate? Prove your answer.

2. Show that a rotation followed by a translation is another rotation. Explicitly, if S is the rotation about the origin by angle θ anti-clockwise, and T is a translation by a vector $\mathbf{b}$, what is the centre of the rotation $T \circ S$? What is the angle? Prove your anwer.
3. Show that the composition of two rotations is another rotation. Explicitly, if S is the rotation about the origin by an angle θ anti-clockwise, and T is the rotation about a point $\mathbf{v}$ by an angle ω, what is the centre of the rotation $T \circ S$? What is the angle? Prove your answer. Hint: you might find the previous question helps.
4. Show that a rotation followed by a reflection about a line passing through the centre of the rotation is a reflection. Explicitly, if S is a rotation about the origin by an angle of θ anti-clockwise, and T is a reflection in the line $y = cx$ for some constant c (we usually take $c = \tan(\omega/2)$), what is line of reflection? Prove your answer. (Hint: you might find the first question helps, though this is harder).

C: Extensions

1. Finish the classification of compositions by looking at the pairs we didn't cover above.
2. Check that the set of motions of the form "rotation followed by translation" is closed under composition.
3. (a) Show that any motion of $\mathbb{E}^n$ is invertible.
   (b) If T is a rotation in $\mathbb{E}^2$, what is the inverse?
   (c) If S is a translation in $\mathbb{E}^2$, what is the inverse?
   (d) If U is a reflection in $\mathbb{E}^2$, what is the inverse?
   (e) With T, S, U as above, what is $S^{-1} \circ T \circ S$? What about $S^{-1} \circ U \circ S$?
   (f) (Extension part, for those who know more group theory) This is showing that the *group of orthogonal matrices* (motions fixing a particular point P) is a *normal subgroup* of the *group of all Euclidean motions*, and in fact the group of all motions is a semi-direct product of the orthogonal motions and the translations. Understand this sentence! (More realistically, come back to this question once you learn what a semi-direct product is).