MA 243 HOMEWORK 9

SOLUTIONS

B: Exercises

(1) **Find a projective transformation of $\mathbb{P}^2$ taking the (ordered) list $\{(1:1:0),(1:0:1),(1:1:1),(0:1:1)\}$ of points to the (ordered) list $\{(1:0:0),(0:1:0),(0:0:1),(1:1:1)\}$.**

$$T([\mathbf{x}]) = \left[\begin{pmatrix} 2 & 0 & -2 \\ 2 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}\mathbf{x}\right].$$

(2) **Compute the cross-ratio $\{P,Q;R,S\}$ of the set $\{P=(1:0),Q=(1:1),R=(2:1),S=(1:2)\}$ of points in $\mathbb{P}^1$.**

The transformation

$$T([\mathbf{x}]) = \left[\begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{x}\right]$$

takes P to $(1:0)$, Q to $(0:1)$, and S to $(1:1)$, and takes R to $(1:-1)$, so $\{P,Q;R,S\} = -1$.

(3) **Recall that we embed $\mathbb{A}^n$ into $\mathbb{P}^n$ by sending $\mathbf{x}$ to $(1:\mathbf{x})$. Given an affine transformation $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$, write down the corresponding projective transformation it extends to (this was given in class briefly). Let $S(\mathbf{x}) = A'\mathbf{x}+\mathbf{b}'$. Write down the composition $S\circ T$, and compare it with the result of composing the corresponding projective transformations.**

We set

$$\tilde{T}([\mathbf{x}]) = \left[\left(\begin{array}{c|c} 1 & 0 \\ \hline \mathbf{b} & A \end{array}\right)\mathbf{x}\right].$$

Let

$$\tilde{S}([\mathbf{x}]) = \left[\left(\begin{array}{c|c} 1 & 0 \\ \hline \mathbf{b}' & A' \end{array}\right)\mathbf{x}\right].$$

Then $S\circ T(\mathbf{x}) = A'A\mathbf{x} + (A'\mathbf{b}+\mathbf{b}')$, so

$$\widetilde{S\circ T}([\mathbf{x}]) = \left[\left(\begin{array}{c|c} 1 & 0 \\ \hline (A'\mathbf{b}+\mathbf{b}') & A'A \end{array}\right)\mathbf{x}\right] = \tilde{S}\circ\tilde{T}([\mathbf{x}]).$$

(4) **Read the Proposition in Section 5.6 of the notes. Suppose that the cross ratio $\{P, Q; R, S\} = \lambda$. There are twenty-four permutations (bijections) $\pi : \{P, Q, R, S\} \to \{P, Q, R, S\}$. How many different values does $\{\pi(P), \pi(Q); \pi(R), \pi(S)\}$ take? Hint: See exercises to Chapter five in the notes.**
The Proposition in Section 5.6 of the notes says that if $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ are the position vectors of P, Q, R, S, then

$$\{P, Q; R, S\} = \frac{\mathbf{p} - \mathbf{r}}{\mathbf{p} - \mathbf{s}} \cdot \frac{\mathbf{q} - \mathbf{s}}{\mathbf{q} - \mathbf{r}},$$

where the ratio of vectors means the ratio (of lengths) along the line L, and thus can be taken to be the ratio of signed lengths such as $\pm|\mathbf{p} - \mathbf{r}|/|\mathbf{p} - \mathbf{s}|$. Thus switching P and Q or switching R and S changes a cross-ratio of λ to one of $1/\lambda$. By direct calculation, if $\{P, Q; R, S\} = \lambda$, then $\{Q, R; S, P\} = \lambda/(\lambda - 1)$. Since the permutations (12) and (1234) generate the permutation group S_4 (exercise!), to compute all the possible cross-ratios we just need to work out the results of repeated application of the maps $\phi : \lambda \mapsto 1/\lambda$ and $\psi : \lambda \mapsto \lambda/(\lambda - 1)$. This gives the options: $\{\lambda, 1/\lambda, \lambda/(\lambda - 1), (\lambda - 1)/\lambda, 1/(1 - \lambda), 1 - \lambda\}$, which are $\{1, \phi, \psi, \phi \circ \psi, \psi \circ \phi, \phi \circ \psi \circ \phi\}$. Check that all other compositions give something on this list. Bonus exercise: What can you say about the map from S_4 to the group generated by ϕ, ψ?

Warning: Note a typo in Exercise 5.11 on page 112 of the notes.