MA 243 HOMEWORK 7

SOLUTIONS

B: Exercises

(1) **Find an affine transformation of $\mathbb{A}^2$ taking the set $\{(3,4),(4,6),(6,11)\}$ to $\{(1,-1),(2,1),(3,5)\}$.**

$$T(\mathbf{x}) = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 \\ -7 \end{pmatrix}.$$

(2) **Recall that an affine subspace of dimension d is a subset of $\mathbb{A}^n$ of the form $\mathbf{v} + V = \{\mathbf{v} + \mathbf{w} : \mathbf{w} \in V\}$ where V is a subspace of dimension d. A collection of d points in $\mathbb{A}^n$ are *affine linearly dependent* if there is an affine subspace of dimension $d-2$ containing them.**

(a) **Are the points $\{(1,0,0),(2,2,3),(5,8,12)\}$ affine linearly dependent? What does it mean geometrically for three points to be affine linearly dependent?** These points are affinely linearly dependent, as they all live in the line $L = \{(1,0,0) + \lambda(1,2,3) : \lambda \in \mathbb{R}\}$. In general, three points are affine linearly dependent if and only if they are collinear.

(b) **Give a determinantal criterion for 3 points in $\mathbb{A}^2$ to be affine linearly dependent (ie describe a matrix whose determinant is zero or nonzero accordingly).** Let the three points have position vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$. Form the 3×3 matrix A whose first column is $(1, x_1, x_2)$, second column is $(1, y_1, y_2)$, and whose third column is $(1, z_1, z_2)$. Then the three points are affinely linearly dependent if and only if $\det(A) = 0$. To see this, note that they are linearly dependent if and only if there is some $\lambda \in \mathbb{R}$ with $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$, so if and only if $(1, \mathbf{z}) = (1-\lambda)(1, \mathbf{x}) + \lambda(\mathbf{y})$, and thus if and only if the columns of A linearly dependent.

(c) **Generalize the previous part to the case of $n+1$ points in $\mathbb{A}^n$.** Form the $(n+1) \times (n+1)$ matrix whose columns are the vectors $(1, \mathbf{x})$ for the $n+1$ points $\mathbf{x}$. The points are affinely linearly dependent if and only if the determinant of this matrix is zero.

(3) **Find the intersection of the following pairs of lines in $\mathbb{P}^2$: $L = W/\sim$, $L' = W'/\sim$, where**

(a) $W = \{\mathbf{x} \in \mathbb{R}^3 : x_0 + x_1 = 0\}, W' = \{\mathbf{x} \in \mathbb{R}^3 : 2x_0 + x_1 - x_2 = 0\}$
This is the point $(1 : -1 : 1)$.

(b) $W = \{\mathbf{x} \in \mathbb{R}^3 : x_2 = 0\}, W' = \{\mathbf{x} \in \mathbb{R}^3 : x_3 = x_2\}$
This the point $(1 : 0 : 0)$.

(4) **Prove that 3 lines L, M, N of $\mathbb{P}^n$ that intersect in pairs are either concurrent (have a common point) or coplanar.**

Suppose that L, M, N are not concurrent. Then in particular the three lines are distinct, and there are points $A, B, C \in \mathbb{P}^n$ such that A is the intersection of L and M, B is the intersection of L and N, and C is the intersection of M and N. Pick lifts $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{n+1}$ of A, B, and C. Let W be the span of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$. Note that $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent, since otherwise the points A, B, and C would all lie on a line L', which would be equal to L, since $A, B \in L'$, and also equal to M, since $A, C \in L'$, and also equal to N, since $B, C \in L'$, contradicting the lines being distinct. Thus the span W of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is three-dimensional, so $W/\sim$ is a plane in $\mathbb{P}^n$. Since $A, B \in W/\sim$, $L \in W/\sim$. Similarly $A, C \in W/\sim$ implies that $M \in W/\sim$, and $B, C \in W/\sim$ implies that $N \in W/\sim$. So L, M, N all live in the plane $W/\sim$, so are coplanar.