MA 243 HOMEWORK 7

DUE: THURSDAY, NOVEMBER 27, BY 12PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

(1) Find an affine transformation of $\mathbb{A}^2$ taking the set $\{(0,0),(1,0),(1,1)\}$ to the set $\{(1,2),(1,3),(2,2)\}$.
(2) Are the points $\{(1,3),(2,4),(3,5)\}$ collinear? How about $\{(1,2),(4,5),(-1,0)\}$?

B: Exercises

(1) Find an affine transformation of $\mathbb{A}^2$ taking the set $\{(3,4),(4,6),(6,11)\}$ to $\{(1,-1),(2,1),(3,5)\}$.
(2) Recall that an affine subspace of dimension d is a subset of $\mathbb{A}^n$ of the form $\mathbf{v}+V = \{\mathbf{v}+\mathbf{w} : \mathbf{w} \in V\}$ where V is a subspace of dimension d. A collection of d points in $\mathbb{A}^n$ are *affine linearly dependent* if there is an affine subspace of dimension $d-2$ containing them.
   (a) Are the points $\{(1,0,0),(2,2,3),(5,8,12)\}$ affine linearly dependent? What does it mean geometrically for three points to be affine linearly dependent?
   (b) Give a determinantal criterion for 3 points in $\mathbb{A}^2$ to be affine linearly dependent (ie describe a matrix whose determinant is zero or nonzero accordingly).
   (c) Generalize the previous part to the case of $n+1$ points in $\mathbb{A}^n$.
(3) Find the intersection of the following pairs of lines in $\mathbb{P}^2$: $L = W/\sim$, $L' = W'/\sim$, where
   (a) $W = \{\mathbf{x} \in \mathbb{R}^3 : x_0 + x_1 = 0\}, W' = \{\mathbf{x} \in \mathbb{R}^3 : 2x_0 + x_1 - x_2 = 0\}$
   (b) $W = \{\mathbf{x} \in \mathbb{R}^3 : x_2 = 0\}, W' = \{\mathbf{x} \in \mathbb{R}^3 : x_1 = x_2\}$
(4) Prove that 3 lines L, M, N of $\mathbb{P}^n$ that intersect in pairs are either concurrent (have a common point) or coplanar.

C: Extensions

(1) We can define affine and projective space over any field k. Affine space $\mathbb{A}_k^n$ is the vector space k^n with affine transformations $T(x) = Ax + b$ where A is an $n \times n$ invertible matrix with entries in k, and $b \in k^n$. Projective space $\mathbb{P}^n$ is $(k^{n+1} \setminus \mathbf{0})/\sim$, where $\sim$ is defined as before: $\mathbf{v} \sim \lambda\mathbf{v}$ for all $\lambda \in k \setminus 0$.
   (a) Consider the case $k = \mathbb{F}_2$, the finite field with two elements. What do lines look like in $\mathbb{A}^2$?
   (b) What about $\mathbb{A}^3$?
   (c) Repeat this for $k = \mathbb{F}_3$, the finite field with three elements.
   (d) Look at the game described at `http://www.setgame.com/set/index.html`. Can you see a connection?
   (e) Let $k = \mathbb{F}_2$. List the points in $\mathbb{P}_k^2$. Draw a picture of all the lines in $\mathbb{P}_k^2$.