MA 243 HOMEWORK 4

SOLUTIONS

B: Exercises

(1) **Prove that if triangles ABC and $A'B'C'$ have $d(A,B) = d(A',B')$, and the angles at A and B equal the angles at A' and B' respectively: $\angle BAC = \angle B'A'C', \angle ACB = \angle A'C'B'$, then ABC is congruent to $A'B'C'$. Use the language and definitions of this module.**

Fix a choice of coordinates so that A is the origin, B lies on the positive x-axis, and C lies in the upper half plane. Let $\{P_0, P_1, P_2\}$ be the Euclidean frame given by setting $P_0 = A'$, P_1 is on the line $A'B'$ in the direction of B', and P_2 lies on the same side of the line $A'B'$ as C'. Let T be the motion that takes the standard frame $\{(0,0), (1,0), (0,1)\}$ to $\{P_0, P_1, P_2\}$. Then $T(A) = A'$ by construction. Since $d(A,B) = d(A',B')$, and the line AB is taken to the line $A'B'$, we have $T(B) = B'$. Since motions preserve angles, and $\angle BAC = \angle B'A'C'$, the line AC is taken to the line $A'C'$. Similarly the line BC is taken to the line $B'C'$. Thus the point C, which is the intersection of the two lines AC and BC, is taken to the intersection of the lines $A'C'$ and $B'C'$, which is C'. So $T(A) = A$, $T(B) = B$, and $T(C) = C'$, so there is a motion taking ABC to $A'B''C'$ and so the two triangles are congruent.

(2) **Let T be the motion of $\mathbb{E}^3$ given in coordinates by $T(x_1, x_2, x_3) = (-x_3, -x_2, x_1)$. Write T as the composition of rotations and reflections and a translation as described in class. Then write T as the composition of at most four reflections as described in class.**

$$T(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is the composition of the reflection in the x_1x_3 plane and the rotation by $3\pi/2$ anti-clockwise about the positive x_2 axis.

To write T as a product of reflections, note that $\mathrm{Fix}(T) = \{0\}$. Let $P_1 = \mathbf{e}_2$, so $Q_1 = T(P_1) = -\mathbf{e}_2$. Let S_1 be the reflection in the plane $x_2 = 0$, which is the perpendicular bisector of P_1Q_1, and let $T_1 = S_1 \circ T$. We then have

$$T_1(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Letting $P_2 = \mathbf{e}_1$, and $Q_2 = T(P_2) = -\mathbf{e}_3$. Let S_2 be the reflection in the plane $x_1 + x_3 = 0$, which is the perpendicular bisector of P_2Q_2, and let $T_2 = S_2 \circ T_1$. Then

$$S_2(\mathbf{x}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

so

$$T_2(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that T_2 is the reflection in the x_1x_2-plane (with equation $x_3 = 0$. So $T = S_1 \circ S_2 \circ T_2$ is the composition of first reflection in the plane $x_3 = 0$, then the plane $x_1 + x_3 = 0$, then the plane $x_2 = 0$.

(3) **Recall that the perpendical bisector of $P, Q \in \mathbb{E}^n$ is the set $\{R \in \mathbb{E}^n : d(P,R) = d(Q,R)\}$. Show that this is an affine hyperplane in $\mathbb{E}^n$.** Choose coordinates so that $P = \mathbf{0}$, and $Q = a\mathbf{e}_1$ for $a > 0$. Given $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $d(P,\mathbf{x}) = \sqrt{\sum_{i=1}^n x_i^2}$, and $d(Q,\mathbf{x}) = \sqrt{(x_1 - a)^2 + \sum_{i=2}^n x_i^2}$, so $d(P,\mathbf{x}) = d(Q,\mathbf{x})$ if and only if $(x_1 - a)^2 = x_1^2$, so $x_1 = a/2$. Thus the affine bisector of P and Q is the affine hyperplane $x_1 = a/2$. This can also be rewritten as $a/2\mathbf{e}_1 + \mathrm{span}(\mathbf{e}_2, \dots, \mathbf{e}_n)$.

(4) **Write an equation for the perpendicular bisector Π of the line between $(2,0,0)$ and $(2,1,3)$. Write down in coordinates the motion of reflecting in the plane Π.** The perpendicular bisector of $(2,0,0)$ and$(2,1,3)$ is

$$\Pi = \{(x,y,z) : y + 3z = 5\}.$$

The reflection in Π is given by

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & -3/5 & -4/5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$