MA 243 HOMEWORK 3

SOLUTIONS

B: Exercises

1. **What is the composition of two reflections? Let T and S be two reflections about lines L and M.**
   (a) **Show that if L and M intersect then the composition $T \circ S$ is a rotation. What is the angle? Prove your answer.** Choose coordinates so that the point of intersection is the origin, and M is the x-axis. Let the angle between L and M be θ. Consider the frame $\{(0,0),(1,0),(0,1)\}$, which is taken by $T \circ S$ to $\{(0,0),(\cos(2\theta),\sin(2\theta)),(\cos(\pi/2+2\theta),\sin(\pi/2+2\theta)) = (-\sin(2\theta),\cos(2\theta)\}$. Thus the effect on the frame of $T \circ S$ is the same as anti-clockwise rotation by 2θ about the origin, so $T \circ S$ is this rotation. In a coordinate-free description, the rotation is about the point of intersection, by twice the angle between the lines, in the direction from M to L.
   (b) **Show that if L and M do not intersect then $T \circ S$ is translation. By what vector does it translate? Prove your answer.** Choose coordinates so that Mx is taken to the x-axis, and L is the line $y = a$ for $a > 0$. Then the frame $\{(0,0),(1,0),(0,1)\}$ is taken to $\{(0,2a),(1,2a),(0,2a+1)\}$. Thus $T \circ S$ is the translation by $(0,2a)$, or in coordinate independent form by a vector perpendicular to M pointing towards L with length twice the distance between L and M.
2. **Show that a rotation followed by a translation is another rotation. Explicitly, if S is the rotation about the origin by angle θ anti-clockwise, and T is a translation by a vector $\mathbf{b}$, what is the centre of the rotation $T \circ S$? What is the angle? Prove your anwer.** In coordinates $T \circ S(\mathbf{x}) = Ax + \mathbf{b}$, where

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

This is equal to

$$T \circ S(\mathbf{x}) = A(\mathbf{x} - \mathbf{u}) + \mathbf{u},$$

for $\mathbf{u} = (1/2b_1 - 1/2b_2 \cot(\theta/2), 1/2b_1 \cot(\theta/2) + 1/2b_2)$, so is rotation by θ anti-clockwise about the point $\mathbf{u}$. To find $\mathbf{u}$ one need only solve the equation $T(\mathbf{x}) = \mathbf{x}$, and use repeatedly the identity $(1 - \cos(\theta))/\sin(\theta) = \tan(\theta/2)$.

3. **Show that the composition of two rotations is another rotation. Explicitly, if S is the rotation about the origin by an angle θ anti-clockwise, and T is the rotation about a point $\mathbf{v}$ by an angle ω, what is the centre of the rotation $T \circ S$? What is the angle? Prove your answer. Hint: you might find the previous question helps.** Write $S(\mathbf{x}) = A_\theta \mathbf{x}$, and $T(\mathbf{x}) = A_\omega(\mathbf{x} - \mathbf{v}) + \mathbf{v}$. Then

$$\begin{aligned} T \circ S(\mathbf{x}) = & \quad A_\omega(A_\theta \mathbf{x} - \mathbf{v}) + \mathbf{v} \\ = & \quad A_{\theta+\omega}\mathbf{x} + (\mathbf{v} - A_\omega \mathbf{v}), \\ = & \quad A_{\theta+\omega}(\mathbf{x} - \mathbf{u}) + \mathbf{u}, \end{aligned}$$

where

$$\mathbf{u} = \begin{pmatrix} (1/2(\mathbf{v} - A_\omega \mathbf{v})_1 - 1/2 \cot((\theta + \omega)/2)(\mathbf{v} - A_\omega \mathbf{v})_2 \\ 1/2(\mathbf{v} - A_\omega \mathbf{v})_1 \cot((\theta + \omega)/2) + 1/2(\mathbf{v} - A_\omega \mathbf{v})_2) \end{pmatrix}$$

4. **Show that a rotation followed by a reflection about a line passing through the centre of the rotation is a reflection. Explicitly, if S is a rotation about the origin by an angle of θ anti-clockwise, and T is a reflection in the line $y = cx$ for some constant c (we usually take $c = \tan(\omega/2)$), what is line of reflection? Prove your answer. (Hint: you might find the first question helps, though this is harder).**

Write $S(\mathbf{x}) = A_\theta \mathbf{x}$, and $T(\mathbf{x}) = B_\omega \mathbf{x}$, where A_θ is the matrix for rotation by θ, and B_ω is the matrix for reflecting in the line

$y = \tan(\omega/2)x$. Then

$$\begin{aligned}
T \circ S(\mathbf{x}) &= B_\omega A_\theta \mathbf{x} \\
&= \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ \sin(\omega) & -\cos(\omega) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mathbf{x} \\
&= \begin{pmatrix} \cos(\omega)\cos(\theta) + \sin(\omega)\sin(\theta) & -\cos(\omega)\sin(\theta) + \sin(\omega)\cos(\theta) \\ \sin(\omega)\cos(\theta) - \cos(\omega)\sin(\theta) & -\sin(\omega)\sin(\theta) - \cos(\omega)\cos(\theta) \end{pmatrix} \mathbf{x} \\
&= \begin{pmatrix} \cos(\omega - \theta) & \sin(\omega - \theta) \\ \sin(\omega - \theta) & -\cos(\omega - \theta) \end{pmatrix} \mathbf{x},
\end{aligned}$$

so $T \circ S$ is the reflection in the line $y = \tan((\omega - \theta)/2)x$.