AARMS TROPICAL GEOMETRY - LECTURES 6 AND 7

DIANE MACLAGAN

The goal for today is to define tropical varieties and state the fundamental theorem of tropical varieties.

As always, K is an algebraically closed field with a nontrivial valuation $\mathrm{val} : K \to \mathbb{R} \cup \infty$. It will never be wrong to take $K = \mathbb{C}\{\{t\}\}$.

We first recall the definition of Gröbner bases in $S = \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ from last time. Technically the definition I gave in the previous lecture used the notation t^{-W}, so only made sense in the Puisuex series field. We first define a notion of t^a for $a \in \mathrm{im(val)}$ for an arbitrary algebraically closed field K with a valuation val.

Lemma 1. *Let K be an algebraically closed field with a valuation* $\mathrm{val} : K \to \mathbb{R} \cup \infty$*, and let* $\mathrm{im(val)}$ *be the additive subgroup of* $\mathbb{R}$ *that is the image of* K^* *under* val*. The surjection of abelian groups* $K^* \twoheadrightarrow \mathrm{im(val)}$ *splits, so there is a group homomorphism* $\phi\colon \mathrm{im(val)} \to K^*$ *with* $\mathrm{val}(\phi(w)) = w$*.*

Proof. Since K is algebraically closed, it contains the nth roots of all of its elements. Thus K^*, and so $\mathrm{im(val)}$ are divisible abelian groups. Since $\mathrm{im(val)}$ is an additive subgroup of $\mathbb{R}$ it is torsionfree, so $\mathrm{im(val)}$ is a torsionfree divisible group, and thus isomorphic to a (possibly uncountable) direct sum of copies of $\mathbb{Q}$ (see, for example, [Hun80, Exercise 8, p198]). Given any summand isomorphic to $\mathbb{Q}$, with $w \in \mathrm{im(val)}$ taken to 1 by the isomorphism, and any $a \in K^*$ with $\mathrm{val}(a) = w$, there is a homomorphism $\phi(\mathbb{Q}) \to K^*$ taking w to K^*. By construction this homomorphism satisfies $\mathrm{val}(\phi(m/nw)) = m/nw$. The universal property of the direct product then implies the existence of a homomorphism $\mathrm{im(val)} \to K^*$ with the desired property. $\square$

We use the notation t^w to denote the element $\phi(w) \in K^*$. We always assume $1 \in \mathrm{im(val)}$, and so $\mathbb{N} \subseteq \mathrm{im(val)}$, so t^n makes sense for any $n \in \mathbb{N}$.

Fix $w \in \mathbb{R}^n$. Given $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in S$, let $W = \min\{\mathrm{val}(c_u) + w \cdot u : c_u \neq 0\}$. Then

$$\mathrm{in}_w(f) = \overline{t^{-W} \sum_{w \in \mathbb{Z}^n} c_u t^{w \cdot u} x^u} \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

and

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

Example: Let $f = 3tx^2 + 5xy + 7ty^2 + 9x + y + 2t$, and let $I = \langle f \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Fix $w = (w_1, w_2) \in \mathbb{R}^2$. Then

$$\mathrm{in}_w(f) = \overline{t^{-W}(3t^{2w_1+1}x^2 + 5t^{w_1+w_2}xy + 7t^{2w_2+1}y^2 + 9t^{w_1}x + t^{w_2}y + 2t)} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}],$$

where

$$W = \min(2w_1 + 1, w_1 + w_2, 2w_2 + 1, w_1, w_2, 1).$$

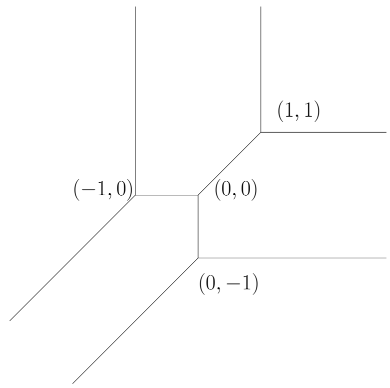

FIGURE 1.

For example, if $W = 2w_1 + 1$ and all other terms are larger, then $\mathrm{in}_w(f) = 3x^2$, and $\mathrm{in}_w(I) = \langle 3x^2 \rangle = \langle 1 \rangle$. So for $\mathrm{in}_w(I) \neq \langle 1 \rangle$, a necessary condition is that the minimum in the definition of W is achieved twice!

For example, if $2w_1 + 1 = w_1 \leq w_1 + w_2, 2w_2 + 1, w_2, 1$, then $w_1 = -1, w_2 \geq 0$. In this case $\mathrm{in}_w(f) = 3x^2 + 9x$, so $\mathrm{in}_w(I) = \langle 3x^2 + 9x \rangle = \langle x + 3 \rangle \neq \langle 1 \rangle$. The set of w for which $\mathrm{in}_w(I) \neq \langle 1 \rangle$ is illustrated in Figure 1.

We now recall from the first day of class the definition of the tropical hypersurface. Given $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we defined

$$\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_u) + w \cdot u : c_u \neq 0),$$

and

$$\mathrm{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{ the minimum in } \mathrm{trop}(f)(w) \text{ is achieved twice } \}.$$

Recall that if $X \subset T^n$ is a variety with (radical) ideal I then $X = \bigcap_{f \in I} \{x \in T^n : f(x) = 0\}$.

Definition 2. Let $X \subseteq T^n$ be a subvariety of T^n with (radical) ideal I. Then the *tropical variety* or *tropicalization* of X is

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)).$$

Warning: In the "classical" world we have

$$X = \bigcap_{f \in \mathcal{G}} V(f)$$

where $\mathcal{G}$ is *any* generating set for the ideal of X. The analogue is not true tropically.
Example: Let $X = V(x+y+1, x+2y+3) \subseteq T^2$. Note that $X = V(y+2, x-1) = \{(1, -2)\} \subseteq T^2$. However $\mathrm{trop}(V(x + y + 1)) = \mathrm{trop}(V(x + 2y + 3)) = \{(u, v) \in \mathbb{R}^2 : u = v \leq 0\} \cup \{(u, v) \in \mathbb{R}^2 : u = 0 \leq v\} \cup \{(u, v) \in \mathbb{R}^2 : v = 0 \leq v\}$ as shown in Figure 2. Thus $\mathrm{trop}(V(x + y + 1)) \cap \mathrm{trop}(V(x + 2y + 3))$ is the union of these three line segments. However $\mathrm{trop}(V(y + 2))$ is the line $y = 0$, while $\mathrm{trop}(V(x - 1))$ is the line $x = 0$, so their intersection is the point $(0, 0)$.

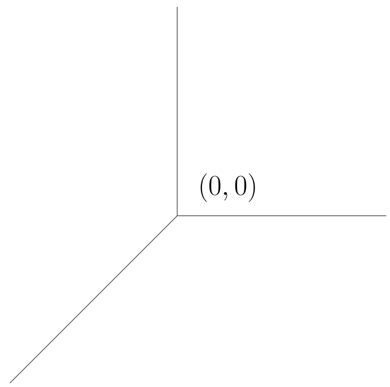

FIGURE 2.

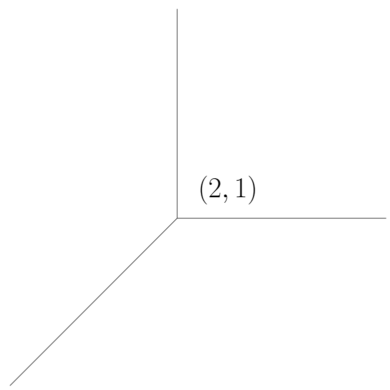

FIGURE 3.

Fundamental Theorem of Tropical Geometry.

Theorem 3. *Let* $X \subseteq T^n_K$ *be a subvariety of* T^n *with (radical) ideal* $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. *Then the following subsets of* $\mathbb{R}^n$ *coincide:*

(1) $\mathrm{trop}(X)$;
(2) $\{w \in \mathbb{R}^n : \mathrm{in}_w(I) \neq \langle 1 \rangle\}$;
(3) *The closure in* $\mathbb{R}^n$ *of* $\{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \in \mathbb{R}^n : x = (x_1, \ldots, x_n) \in X\}$.

We will sketch a proof of this theorem below, starting in the hypersurface case (when $I(X)$ is a principal ideal). We first illustrate the theorem with an example.
Example: Let $X = V(f) \subseteq T^2_K$ for $f = tx + 3t^2y + t^3 \in K[x^{\pm 1}, y^{\pm 1}]$, and let $I = \langle f \rangle$. Here, as always if it is not otherwise indicated, we take $K = \mathbb{C}\{\{t\}\}$. The three sets of Theorem 3 are constructed as follows.

(1) We have $\mathrm{trop}(f) = \min(x+1, y+2, 3)$, so the set $\mathrm{trop}(V(f)) = \{(u,v) \in \mathbb{R}^2 : u = 2 \leq v\} \cup \{(u,v) \in \mathbb{R}^2 : v = 1 \leq u\} \cup (u,v) \in \mathbb{R}^2 : u = v+1 \leq 2\}$. This is illustrated in Figure 3.

We are using here that the two possible definitions of $\mathrm{trop}(V(f))$ coincide, so the set where the minimum in the definition of $\mathrm{trop}(f)$ is achieved twice is equal to the intersection over *all* $g \in I$ of the set where the minimum in the definition of $\operatorname{trop}(g)$ is achieved twice. This is an exercise in the second exercise set.

(2) Given $w \in \mathbb{R}^2$, the initial term $\operatorname{in}_w(f)$ is the image in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ of

$$t^{-W}(t^{w_1+1}x + 3t^{w_2+2}y + t^3),$$

where $W = \min(w_1+1, w_2+2, 3)$. If this minimum is achieved only once then $\operatorname{in}_w(f)$ is a monomial, so $\operatorname{in}_w(I) = \langle 1 \rangle$. So if $\operatorname{in}_w(I) \neq \langle 1 \rangle$, then the minimum is achieved at least twoice. If the minimum is achieved at least twice, then $\operatorname{in}_w(f)$ is not a monomial. It is

$$\begin{aligned} x + 3y &\text{ if } w_1 + 1 = w_2 + 2 < 3; \\ x + 1 &\text{ if } w_1 + 1 = 3 < w_2 + 2; \\ 3y + 1 &\text{ if } w_2 + 2 = 3 < w_1 + 1; \\ x + 3y + 1 &\text{ if } (w_1, w_2) = (2, 1). \end{aligned}$$

Thus, since $\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) \rangle$, we have

$$\{w \in \mathbb{R}^2 : \operatorname{in}_w(f) \neq \langle 1 \rangle\} = \operatorname{trop}(X).$$

(3) The variety X is

$$\begin{aligned} X &= \{(x, y) \in T^2_K : tx + 3t^2y + t^3 = 0\} \\ &= \{(-t^2 - 3ty, y) : y \in K^*, 3ty + t^2 \neq 0\}. \end{aligned}$$

Thus

$$\begin{aligned} &\overline{(\operatorname{val}(x), \operatorname{val}(y)) : (x, y) \in X\}} \\ &= \overline{\{(\operatorname{val}(-t^2 - 3ty), \operatorname{val}(y)) : y \in K^*, y \neq -t/3\}}. \end{aligned}$$

Now $\operatorname{val}(-t^2 - 3ty) = \min(2, \operatorname{val}(y) + 1)$ if $y \neq -t/3 + z$ with $\operatorname{val}(z) > 1$. Thus $\overline{(\operatorname{val}(x), \operatorname{val}(y)) : (x, y) \in X\}} = \{(2, w) : w \geq 1\} \cup \{(w+1, w) : w \leq 1\} \cup \{(w, 1) : w \geq 2\}$. So all three sets coincide in this example.

We now begin the proof of Theorem 3. We first prove this is in the hypersurface case.

Proposition 4. *Let K be an algebraically closed field with a nontrivial valuation* val*, and let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the following three sets coincide.*

1. $\operatorname{trop}(V(f))$*;*
2. *The set $\{w \in \mathbb{R}^n : \operatorname{in}_w(f)$ is not a monomial $\}$.*
3. *The closure of the set $\{(\operatorname{val}(v_1), \ldots, \operatorname{val}(v_n)) : \mathbf{v} \in T^n_K, f(\mathbf{v}) = 0\}$;*

Proof. Let $(w_1, \ldots, w_n) \in \operatorname{trop}(V(f))$. Then by definition the minimum $W = \min(\operatorname{val}(c_u) + \mathbf{u} \cdot w : c_u \neq 0)$ is achieved at least twice. This then means that $\operatorname{in}_w(f) = \overline{c_u} x^u$ is not a monomial, and thus set (1) is contained in set (2). Conversely, if $\operatorname{in}_w(f)$ is not a monomial, then $\min(\operatorname{val}(c_u) + \mathbf{u} \cdot w : c_u \neq 0)$ is achieved at least twice, so $w \in \operatorname{trop}(V(f))$. This shows the other containment, so the first two sets are equal.

We now prove the inclusion of set (3) in set (1). Since set (1) is closed, it is enough to consider points in (3) of the form $\operatorname{val}(v) := (\operatorname{val}(v_1), \dots, \operatorname{val}(v_n))$ where $\mathbf{v} = (v_1, \dots, v_n) \in T_K^n$ satisfies $f(\mathbf{v}) = 0$. Let $\mathbf{v} \in T_K^n$ satisfy $f(\mathbf{v}) = 0$, so $\sum_{\mathbf{u} \in \mathbb{Z}^n} c_u \mathbf{v}^{\mathbf{u}} = 0$. We first reduce to the case where $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}})) \geq 0$ for all $\mathbf{u}$, so $c_u \mathbf{v}^{\mathbf{u}} \in R$. Let $W = \min\{\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) : c_u \neq 0\}$, and let $g = t^{-W} f$. Then $g(\mathbf{v}) = 0$, and $\operatorname{trop}(V(g)) = \operatorname{trop}(V(f))$, so it suffices to prove the inclusion with f replaced by g. We can thus assume ($\star$) that $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) \geq 0$, and that there is at least one $\mathbf{u}$ with $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) = 0$. Then $f(\mathbf{v}) = \sum c_u \mathbf{v}^{\mathbf{u}} = 0$ is the sum of elements of R, and so we can consider their image in the residue field $\Bbbk = R/\mathfrak{m}$. This is $\sum \overline{c_u \mathbf{v}^{\mathbf{u}}} = 0 \in \Bbbk$. By assumption $\star$ at least one of the terms $\mathbf{v}^{\mathbf{u}}$ is nonzero. Since the sum of all such terms is $0 \in \Bbbk$, we conclude that there must in fact be at least two terms with $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) = 0$. We have $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) = \operatorname{val}(c_u) + \sum u_i \operatorname{val}(v_i) \geq 0$ for all u by assumption $\star$, so this means that the minimum $0 = \min(\operatorname{val}(c_u) + \mathbf{u} \cdot \operatorname{val}(\mathbf{v})) = \min(\operatorname{val}(c_u)) + \mathbf{u} \cdot \operatorname{val}(\mathbf{v}))$ is achieved twice, where $\operatorname{val}(\mathbf{v}) = (\operatorname{val}(v_1), \dots, \operatorname{val}(v_n))$. Thus $\operatorname{val}(\mathbf{v}) \in \operatorname{trop}(V(f))$ as required.

Finally, we prove the inclusion of set (1) into set (3). Since the image of the valuation val is dense in $\mathbb{R}$ (see Exercises), and the set (3) is closed by definition, it suffices to consider a point in (1) of the form $w = \operatorname{val}(\mathbf{y})$ for some $\mathbf{y} \in T_K^n$. We want to construct $\beta = (\beta_1, \dots, \beta_n) \in T_K^n$ with $\operatorname{val}(\beta) = (\operatorname{val}(\beta_1), \dots, \operatorname{val}(\beta_n)) = w$ and $f(\beta) = 0$.

Let $\operatorname{in}_w(f) = \sum a_u x^u \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Since $\operatorname{in}_w(f)$ is not a monomial, there is some variable x_i that appears to different powers in those x^u with $a_u \neq 0$. After reordering, we can assume that this is x_1. Replacing f by $x_1^m f$ for some $m \in \mathbb{Z}$ does not change $\operatorname{trop}(V(f))$ or whether $\operatorname{in}_w(f)$ is a monomial, so we may assume that x_1 appears in some, but not all, of the monomials occuring in $\operatorname{in}_w(f)$, and that if x_1 occurs, the exponent is positive.

Since $\Bbbk$ is algebraically closed and $\operatorname{in}_w(f)$ is not a monomial, we can find $\alpha \in (\Bbbk^*)^n$ with $\operatorname{in}_w(f)(\alpha) = 0$. Let $\beta_i = \alpha_i t^{w_i}$ for $2 \leq i \leq n$. Let $g(y) = f(y, \beta_2, \dots, \beta_n) \in K[y^{\pm 1}]$. The assumption that the exponent of x_1 in every monomial of f is non-negative and that some such exponents are zero means that in fact $g \in K[y]$ is a polynomial of positive degree with nonzero constant term. Since K is algebraically closed we can thus factor g into linear factors:

$$g = \lambda \prod_{i=1}^{m} (y - b_i).$$

Write $\mathbf{u}' = (u_2, \dots, u_n) \in \mathbb{Z}^{n-1}$ for the projection of $\mathbf{u}$ onto the last $n-1$ components. Then $g(y) = \sum_{\mathbf{u} \in \mathbb{Z}^n} (c_u \beta^{\mathbf{u}'}) y^{u_1}$. Note that $\operatorname{val}(\beta^{\mathbf{u}'}) = \sum_{i=2}^n u_i \operatorname{val}(b_i) = \sum_{i=2}^n u_i w_i$, so $\operatorname{val}(c_u \beta^{\mathbf{u}'}) + w_1 u_1 = \operatorname{val}(c_u) + w \cdot \mathbf{u}$.

Thus $\operatorname{in}_{w_1}(g)(y) = \sum_{\mathbf{u}: \operatorname{val}(c_u) + w \cdot \mathbf{u} = W} \overline{t^{w_1 u_1 - W} c_u \beta^{\mathbf{u}'}} y^{u_1} = \sum_{\mathbf{u}: \operatorname{val}(c_u) + w \cdot \mathbf{u} = W} a_u \alpha^{\mathbf{u}'} y^{u_1}$, and so $\operatorname{in}_{w_1}(g)(\alpha_1) = \sum a_u \alpha^{\mathbf{u}} = 0$.

Now $\operatorname{in}_{w_1}(g) = \overline{t^{-\operatorname{val}(\lambda)} \lambda} \operatorname{in}_w(y - b_1) \cdots \operatorname{in}_w(y - b_m)$ (see Exercises). Thus there is some j for which $\operatorname{in}_w(y - b_j)(\alpha_1) = 0$. For this j we must have $\operatorname{val}(b_j) = w_1$, as otherwise $\operatorname{in}_w(a_j y - b_j)$ is a monomial, and so then $\operatorname{in}_w(a_j y - b_j)(\alpha_1) \neq 0$ (since $\alpha_1 \neq 0$ because g has a nonzero constant term). Let $\beta_1 = b_j$. Then $g(\beta_1) = 0$ by construction. Thus if $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, we have $f(\beta) = 0$ and $\mathrm{val}(\beta) = w$, so β is the desired point. $\square$

The proof of Theorem 3 in the general (non-hypersurface) case proceeds by reduction to the hypersurface case, as we now sketch.

Sketch of proof of Theorem 3. The equality of the first two sets is immediate from Proposition 4, as $w \in \mathrm{trop}(X) = \cap_{f \in I} \mathrm{trop}(f)$ if and only if $\mathrm{in}_w(f)$ is not a monomial for all $f \in I$, which occurs if and only if $\mathrm{in}_w(I) \neq \langle 1 \rangle$.

The inclusion of the third set into the first is also an immediate corollary of Proposition 4. If $w = \mathrm{val}(y)$ for $y \in X$, then $f(y) = 0$ for all $f \in I$, so $w \in \mathrm{trop}(V(f))$ for all $f \in I$, so $w \in \mathrm{trop}(X)$.

We are left with showing that the second set is contained in the third set. The key idea is to project X to a hypersurface. We can (proof skipped) assume that X is irreducible of dimension d. We then claim (proof skipped) that for a generic choice of projection $\phi : T^n \to T^{d+1}$ the image $\phi(X)$ is a hypersurface in T^{d+1}. Applying the map $\mathrm{Hom}(K^*, -)$ to the projection $\phi : T^n \to T^{d+1}$ we get a map $\psi : Z^n \to Z^{d+1}$. We write ϕ for both the projection $T^n_K \to T^{d+1}_K$ and for the corresponding projection $T^n_{\Bbbk} \to T^{d+1}_{\Bbbk}$, and also for the maps of coordinate rings $\Bbbk[y_1^{\pm 1}, \dots, y_{d+1}^{\pm 1}] \to \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. For a generic projection we have

$$\phi(\mathrm{in}_w(I))) = \mathrm{in}_{\psi(w)}(\phi(I)).$$

Thus if w lies in the second set of the Theorem, so $\mathrm{in}_w(I) \neq \langle 1 \rangle$, then $\phi(\mathrm{in}_w(I)) = \mathrm{in}_{\psi(w)}(\phi(I)) \neq \langle 1 \rangle \subseteq \Bbbk[y_1^{\pm 1}, \dots, y_{d+1}^{\pm 1}]$. Applying Proposition 4 we see that $\psi(w) \in \mathrm{trop}(\phi(X))$, so there is $y \in \phi(X)$ with $\mathrm{val}(y) = \psi(w)$. Since $y \in \phi(X)$ there is $\tilde{y} \in X$ with $\phi(\tilde{y}) = y$, and $\psi(\mathrm{val}(\tilde{y})) = \psi(w)$. We claim (proof skipped) that we can choose $\tilde{y}$ with $\mathrm{val}(\tilde{y}) = w$, which shows that w lies in the third set of the theorem. $\square$

Remark 5. Theorem 3 says that if $w \in \mathrm{trop}(X)$ then there is $y \in X$ with $\mathrm{val}(y) = w$. Sam Payne has shown that the set

$$\{y \in X : \mathrm{val}(y) = w\}$$

is Zariski dense in X for all $w \in \mathrm{trop}(X)$. See [Pay07] for details.

We finish this lecture with examples of tropical varieties for which the classical variety is not a hypersurface.

Example: Let $X = V(x_1 + x_2 + x_3 + 1, x_2 + 2x_3 + 3\rangle \subset T^3$. Then

$$\begin{aligned} X &= V(x_1 - x_3 - 2, x_2 + 2x_3 + 3\rangle \\ &= \{(2+s, -3-2s, s) : s \in K^* : t \neq -2, -3/2\}. \end{aligned}$$

Now

$$\mathrm{val}(2+, -3-2s, s) = \begin{cases} (0, 0, \mathrm{val}(s)) & \text{if } \mathrm{val}(s) > 0 \\ (\mathrm{val}(s), \mathrm{val}(s), \mathrm{val}(s)) & \text{if } \mathrm{val}(s) < 0 \\ (w, 0, 0) & \text{if } s = -2 + s', \mathrm{val}(s') = w > 0 \\ (0, w, 0) & \text{if } s = -3/2 + s', \mathrm{val}(s') = w > 0 \\ (0, 0, 0) & \text{if } \mathrm{val}(s) = 0, \overline{s} \neq -2, -3/2 \end{cases}$$

Thus $\operatorname{trop}((X)$ is the union of the rays through $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$ in $\mathbb{R}^3$.

Example: Let $I = \langle x_1 + x_2 + x_3 + x_4 + 1, x_2 + 2x_3 + 3x_4 + 4\rangle \subseteq \Bbbk[x_1^{\pm 1}, \dots, x_4^{\pm 1}]$, and let $X = V(I) \subseteq T_K^4$. Then $\operatorname{trop}(X)$ is the two-dimensional fan in $\mathbb{R}^4$ with

$$\{(1,0,0,0),(0,1,0,0,),(0,0,1,0),(0,0,0,1),(-1,-1,-1,-1)\}$$

and two-dimensional cones spanned by any two of these. The intersection of $\operatorname{trop}(X)$ with the sphere $S^3 \subseteq \mathbb{R}^4$ is then a graph with five vertices and ten edges (the complete graph K_5).

It is hard to draw pictures of tropical varieties that do not lie in $\mathbb{R}^2$. For two-dimensional tropical varieties we will often resort to this trick of intersecting with the sphere and drawing the corresponding graph.

References

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[Pay07] Sam Payne, *Fibers of tropicalization*, 2007. arXiv:0705.1732. To appear in Math. Z.