AARMS TROPICAL GEOMETRY - LECTURES 17 AND 18

DIANE MACLAGAN

In these last two lectures we describe how to use tropical techniques to count rational curves in $\mathbb{P}^2$ passing through a fixed number of points.

Definition 1. A curve $C = V(f) \subset \mathbb{P}^2$ has degree d if $f \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree d. An irreducible curve C is rational if there is a map from an open set $U \subset \mathbb{P}1$ to C given by

$$\phi([t_0 : t_1]) = (p_0(t_0, t_1) : p_1(t_0, t_1) : p_2(t_0, t_2))$$

where p_i is a homogeneous polynomial of degree d for $0 \leq i \leq 2$. The open set $U \subset \mathbb{P}^1$ is the set of $[t_0 : t_1]$ for which at least one of $p_0(t_0, t_1)$, $p_1(t_0, t_1)$, and $p_2(t_0, t_1)$ are nonzero.

A curve of degree one is a line in $\mathbb{P}^2$, which is rational.
Exercise: Check that every curve in $\mathbb{P}^2$ of degree two is rational.
Question: Given n general points $p_1, \dots, p_n \in \mathbb{P}^2$, how many rational curves of degree d pass through all n points?

This question clearly needs to be clarified before a clear answer can be given. For small n (such as $n = 1$) there will be an infinite number of curves. For example, there are infinitely many lines through any given point in $\mathbb{P}^2$. For large n if the p_i are not chosen specially, then there are no curves. For example, if p_1, p_2, p_3 are three lines in $\mathbb{P}^2$ that do not lie on a line, then there are no curves of degree one (lines!) passing through all three points. However given any two distinct points in $\mathbb{P}^2$, there is a unique line passing through them, so for $d = 1$ and $n = 2$, with the notion of "general" being "distinct", the question has answer one.

When $d = 2$, we claim that there is also a unique curve of degree two passing through five general points in $\mathbb{P}^2$. To see this, let

$$F = ax_0^2 + bx_0x_1 + cx_0x_1 + dx_1^2 + ex_1x_2 + fx_2^2.$$

The variety $C = V(F)$ is a curve as long as one of $a, \dots, f$ is nonzero, and F and λF define the same curve, so a choice of conic corresponds to a point $[a : b : c : d : e : f] \in \mathbb{P}^5$. We observed in the exercise above that all curves of the form $V(F)$ are rational, so we need to show that given five sufficiently general points in $\mathbb{P}^2$ there is a unique point in $\mathbb{P}^5$ for which the corresponding $V(F)$ passes through all five points. Since we are requiring that the five points be general, we may assume that the first coordinate is nonzero, so they have the form $\mathcal{P}_5 = \{[1 : u_i : v_i] : 1 \leq i \leq 5\}$. So we need the equation

$$\begin{pmatrix} 1 & u_1 & v_1 & v_1^2 & u_1v_1 & v_1^2 \\ 1 & u_2 & v_2 & v_2^2 & u_2v_2 & v_2^2 \\ 1 & u_3 & v_3 & v_3^2 & u_3v_3 & v_3^2 \\ 1 & u_4 & v_4 & v_4^2 & u_4v_4 & v_4^2 \\ 1 & u_5 & v_5 & v_5^2 & u_5v_5 & v_5^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = 0$$

to have a one-dimensional solution space for most choices of $\mathcal{P}_5$. This means that the 5×6 coefficient matrix must have rank five for most choices of five points. The ideal of 5×5 minors of the coefficient matrix is principal, and generated by a single polynomial of degree six in $\mathbb{C}[u_1, \dots, u_5, v_1, \dots, v_6]$, so for any choice of $\mathcal{P}_5$ with this polynomial nonzero there is a unique rational curve of degree two passing through the points in $\mathcal{P}_5$.

Note that this means that for a general set of four points in $\mathbb{P}^2$ there are are an infinite number of degree two rational curves passing through the points, and for a general set of six points in $\mathbb{P}^2$ there are no degree two rational curves passing through all of the points.

Claim: For general d, there are a finite number of rational curves of degree d passing through $3d-1$ general points in $\mathbb{P}^2$.

Idea of proof: A rational curve of degree d in $\mathbb{P}^2$ is determined by three polynomials p_0, p_1, p_2, which have the form $\sum_{j=0}^{d} a_{ij} t_0^j t_1^{d-j}$ for $0 \leq i \leq 2$, for a total of $3d+3$ parameters. Since the image is in $\mathbb{P}^2$, this over-counts by one. Since we only care about the image of the curve, not the parameterization, this also over-counts by the dimension of $\mathrm{Aut}\mathbb{P}^1$, which is three. This means that the set of rational curves is determined by $3d+3-1-3 = 3d-1$ parameters. Thus forcing the curve to pass through $3d-1$ general points in $\mathbb{P}^2$ will guarantee a finite number of solutions.

Definition 2. Let N_d be the number of irreducible rational curves passing through $3d-1$ general points in $\mathbb{P}^2$.

Example: We saw above that $N_1 = N_2 = 1$. The numbers N_3 and N_4 were computed in the nineteenth century.

Theorem 3 (Kontsevich)**.** *For $d > 1$ the numbers N_d obey the following recursion:*

$$N_d = \sum_{d_A + d_B = d, d_A, d_B > 0} (d_A^2 d_B^2 \binom{3d-4}{3d_A-2} - d_A^3 d_B \binom{3d-4}{3d_A-1}) N_{d_A} N_{d_B}.$$

Noe that this describes N_d in terms of smaller d, so knowing $N_1 = 1$ determines all larger N_d.

Example: To compute N_2, the only decomposition is $d_A = d_B = 1$. Then

$$N_2 = (1^2 1^2 \binom{2}{1} - 1^3 1 \binom{2}{2})(1)(1) = 2 - 1 = 1.$$

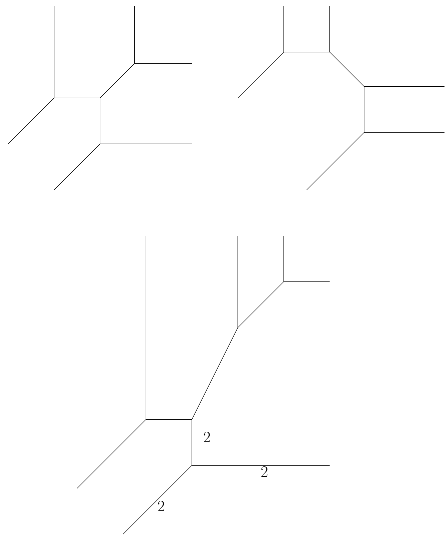

FIGURE 1.

To compute N_3 we need to consider the pairs $(d_A, d_B) \in \{(1,2),(2,1)\}$. Thus

$$\begin{aligned} N_3 =& (1^2 2^2 \binom{5}{1} - 1^3(2)\binom{5}{2})(1)(1) \\ &+ (2^2 1^2 \binom{5}{4} - 2^3(1)\binom{5}{4})(1)(1) \\ =& 20 - 20 + 20 - 8 \\ =& 12 \end{aligned}$$

Tropical Version

We now outline how to prove Theorem 3 using tropical methods. Our sketch follows closely the version given in [HM06], which is based on [GM08] and [Mik05]. The idea is to define a tropical analogue N_d^{trop} of N_d, and show that this equals N_d. We then show that N_d^{trop} obeys the Kontsevich recursion Theorem 3.

Definition 4. A tropical rational curve of degree d is a one-dimensional balanced weighted polyhedral complex for which

(1) the unbounded rays point in the directions $(1,0)$, $(0,1)$, and $(-1,-1)$;
(2) there are d unbounded rays pointing in each of these directions (counted with multiplicity);
(3) and the underlying graph of the polyhedral complex has no cycles.

Example:

Figure 1 shows contains some examples of tropical rational curves of degrees two, two, and three. Figure 2 contains some one-dimensional tropical varieties that are not tropical rational curves of degree d for some d. For the first variety, the problem

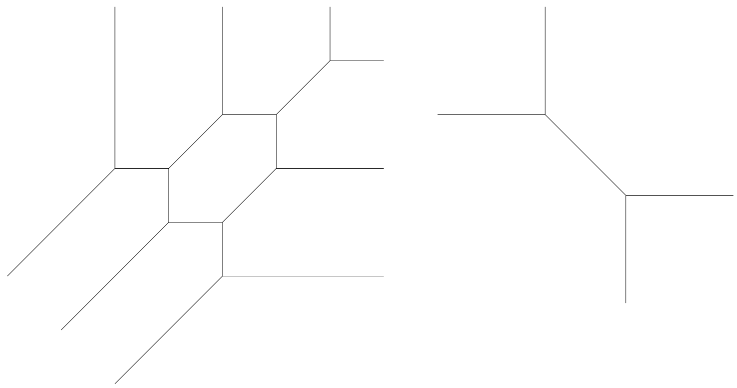

FIGURE 2.

is that the underlying graph contains a cycle. For the second, there are unbounded edges not pointing in one of the three prescribed directions.

Note that there is a unique tropical rational curve of degree one through two points in $\mathbb{R}^2$ unless they lie on the same vertical, horizontal, or slope-one line.

Exercise: There is a unique tropical rational curve of degree two through most sets of five points in $\mathbb{R}^2$.

One way to construct a tropical rational curve of degree d is to take a curve of the form $X = V(f) \subset T^2_{\mathbb{C}\{\{t\}\}}$, where $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ has the form $\sum_{i+j \leq 2, i,j \geq 0} c_{ij} x^i y^j$ with all $c_{ij} \neq 0$. Then, as computed in the first exercise set, $\operatorname{trop}(X)$ is a weighted balanced polyhedral complex with d unbounded rays pointing in the prescribed directions. In [DES07] Speyer shows that every tropical rational curve of degree d in $\mathbb{R}^2$ arises in this fashion.

Definition 5. A rational tropical curve of degree d is trivalent if the underlying graph is trivalent. Let C be a rational tropical curve of degree d, and let V be a vertex of C. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the smallest integral vectors pointing along the three rays leaving V, and let μ_1, μ_2, μ_3 be the multiplicities of the corresponding polyhedra. The *multiplicity* of V is the absolute value of the determinant of the 2×2 matrix with columns two of the $\mu_i \mathbf{v}_i$. The balancing condition guarantees that this is independent of the choice. The *multiplicity* of C is the product of the multiplicity of all vertices in C.

Example : The multiplicity of the first two tropical rational curves shown in Figure 3 is one. The multiplicity of the third curve is $8 = (2)(4)$, since the multiplicity of the bottom vertex is 4, and the multiplicity of the vertex above is 2, while the other two vertices have multiplicity one.

Definition 6. Fix $\mathcal{P} = \{p_1, \dots, p_{3d-1}\} \subset \mathbb{R}^2$. Then $N_d^{trop}(\mathcal{P})$ is the number of tropical rational curves counted with multiplicity passing through $p_1, \dots, p_{3d-1}$.

Proposition 7. *There is a Zariski-open set $U \subset (\mathbb{R}^2)^{3d-1}$ for which $N_d^{\mathrm{trop}}(\mathcal{P})$ is constant for $\mathcal{P} = \{p_1, \dots, p_{3d-1}\}$ with $(p_1, \dots, p_{3d-1}) \in U$.*

Definition 8. A set $\mathcal{P} = \{p_1, \dots, p_{3d-1}\}$ with $(p_1, \dots, p_{3d-1}) \in U$ for the set U of Proposition 7 is said to be in tropical general position. Let $N_d^{trop} = N_d^{trop}(\mathcal{P})$ for any $\mathcal{P}$ in tropical general position

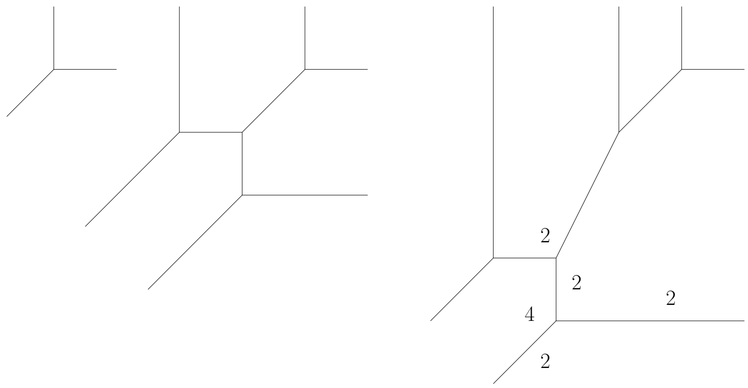

FIGURE 3.

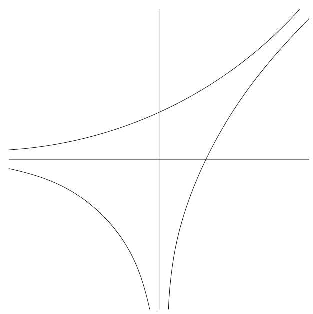

FIGURE 4.

Theorem 9 (Mikhalkin Correspondence Theorem). *We have equality*

$$N_d^{trop} = N_d.$$

The idea of the proof of Theorem 9 is as follows.

The *amoeba* of a variety $X \subset (\mathbb{C}^*)^2$ is the set $\{(\log(|x_1|, \log(|x_2|)) : (x_1, x_2) \in X\} \subset \mathbb{R}^2$.

Example: Let $C = V(x_1 + x_2 + 1) \subset T^2_{\mathbb{C}} = \{(a, -1-a) : a \in \mathbb{C}^*, a \neq -1\}$. Then the amoeba of C is shown in Figure 4.

When $C \subset T^2_{\mathbb{C}}$ has higher degree d, its amoeba generically has d "tentacles" pointing in each direction.

If we replace log by $\log_t$ in the definition of the amoeba, and let $t \to \infty$, then the amoeba gets thinner and thinner, and in the limit approaches a tropical curve of degree d. Given a set of $3d-1$ points we can count the number of rational curves of degree d passing through these points, or the number of tropical rational curves of degree d passing through (roughly) the logs of the points. The multiplicity of a tropical rational curve counts how many of these rational curves in T^2 limit to the tropical curve.

By Theorem 9, in order to prove Theorem 3, it suffices to show that the numbers N_d^{trop} satisfy the same recursion. We show this by using the standard enumerative

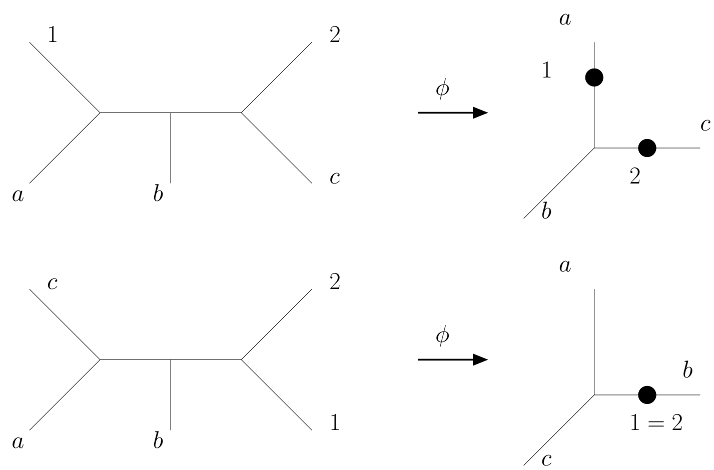

FIGURE 5.

combinatorics trick of arguing that the equation
(1)
$$N_D + \sum_{d_A+d_B=d, d_A,d_B>0} d_A^3 d_B \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B} = \sum_{d_A+d_B=d, d_A,d_B>0} d_A^2 d_B^2 \binom{3d-4}{3d_A-2} N_{d_A} N_{d_B}$$
counts the same objects in two different ways. These objects are *parameterized tropical rational curves* with $n = 3d$ marked points.

Definition 10. A parameterized tropical rational curve of degree d with n marked points is a map $\phi \colon \Gamma \to \mathbb{R}^2$ where Γ is a tree with $3d + n$ leaves such that

(1) n of the leaves of Γ are labelled $1, \dots, n$,
(2) each non-leaf edge e of Γ comes with a weight d_e,
(3) $\phi(\Gamma)$ is a rational tropical curve of degree d,
(4) the image of the labelled leaves of Γ are contracted to points, and
(5) the images of the nonlabelled edges of Γ extended to unbounded rays.

Example: Two examples of parameterized rational curves of degree one with two marked points are shown in Figure 5. We have labelled the "unlabelled" edges a, b, and c so their image under ϕ is clear, but this is not part of the data of ϕ. Note that a non-leaf edge of Γ is contracted in the second example.

Definition 11. Given a parameterized tropical curve of degree d with $n \geq 4$ marked points $\phi : \Gamma \to \mathbb{R}^2$, we define the forgetful map f as follows. Let Γ' be the smallest connected subtree of Γ containing the leaves $1, 2, 3$ and 4. Then Γ' consists of two pairs of leaves which are connected by a path of non-leaf edges. The image of the forgetful map $f(\phi)$ is then the phylogenetic tree $\overline{\Gamma}$ with four leaves obtained by turning this path of edges into one edge with length the combined weights.

Example: An example of the forgetful map applied to a parameterized tropical curve of degree one is shown in Figure 6.

We will count the number $M_d(p_1, \dots, p_{3d-1}; \overline{\Gamma})$ of parameterized tropical rational curves $\phi : \Gamma \to \mathbb{R}^2$ of degree d with $n = 3d$ marked points such that

(1) The labelled edges $3, \dots, n$ get mapped to $p_1, \dots, p_{3d-1}$;

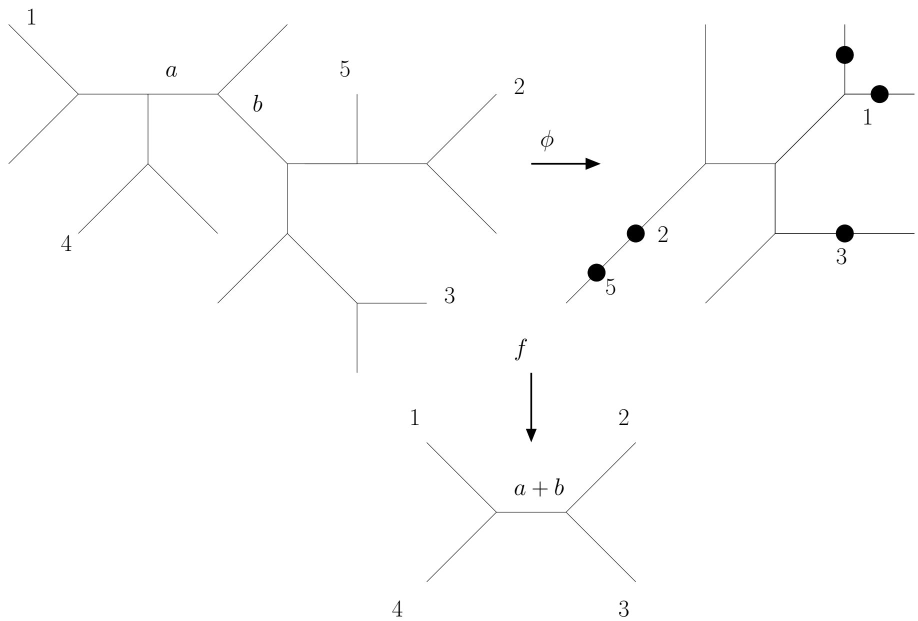

FIGURE 6.

(2) The labelled edge 1 gets mapped to the a point on the line $x = (p_1)_1$;
(3) The labelled edge 2 gets mapped to the a point on the line $y = (p_1)_2$;
(4) The image $f(\phi)$ of the forgetful map is equal to $\overline{\Gamma}$.

Proposition 12. *The number $M_d(p_1, \ldots, p_{3d-1}; \overline{\Gamma})$ is constant for a general choice of $p_1, \ldots, p_{3d-1} \in \mathbb{R}^2$ (tropical general position) and for any choice of four-vertex phylogenetic tree $\overline{\Gamma}$. We thus denote it by M_d.*

The idea of the proof is then to choose $p_1, \ldots, p_{3d-1}$ in tropical general position, and then choose two different trees $\overline{\Gamma}$ at which to evaluate M_d to obtain equation 1.

The proof will use the tropical version of Bézout's theorem. Recall that Bézout's theorem in the plane says that if the variety $V(f_1, f_2) \subset \mathbb{P}^2$ is finite, when f_1, f_2 are homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ of degree d_1 and d_2 respectively, then $V(f_1, f_2)$ consists of $d_1 d_2$ points, counted with multiplicity. The (weak) tropical version of this states that if C_1, C_2 are two tropical curves in $\mathbb{R}^2$, of degrees d_1 and d_2 respectively, that intersect in a finite number of points, then that intersection consists of $d_1 d_2$ points counted with multiplicity. An example of two tropical rational curves of degree two intersecting in four points is shown in Figure 7.

We can now outline the proof of Equation 1. First choose a phylogenetic tree $\overline{\Gamma}$ with four labelled leaves that looks like the one on the left of Figure 8, with the length a of the bounded edge large. Let $\phi : \Gamma \to \mathbb{R}^2$ be a parameterized rational tropical curve of degree d with $n = 3d$ labelled points whose image under the forgetful map is $\overline{\Gamma}$. Then one of two situations occur. The first is that the two leaves labelled 1 and 2 are adjacent to the same vertex in Γ, so the images of these two leaf edges in $\mathbb{R}^2$ is the same. This means that $\phi(\Gamma)$ is a tropical rational curve of degree d in $\mathbb{R}^2$ with $\phi(1) = \phi(2) = p_1$, and $\phi(i) = p_{i-1}$ for $3 \leq i \leq n$. The number of such images is N_d. Such maps ϕ are determind by their image, so there are N_d of such ϕ.

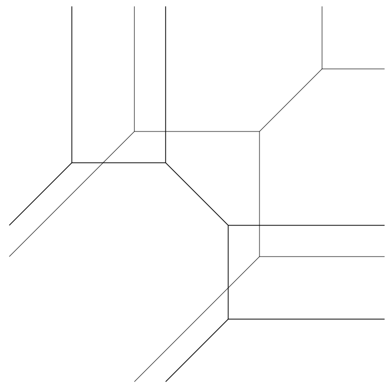

FIGURE 7.

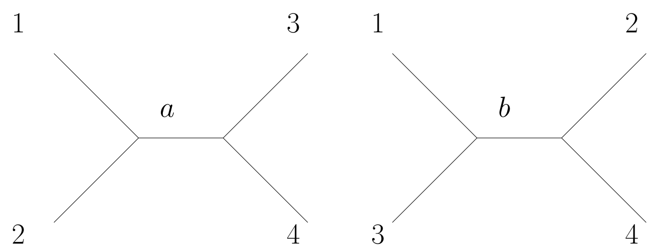

FIGURE 8.

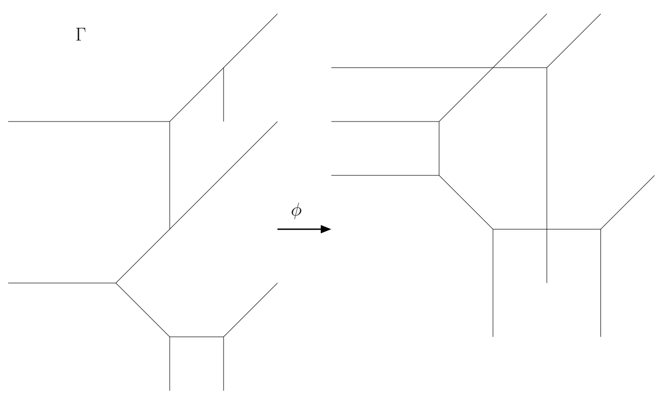

FIGURE 9.

The second possibility is that 1 and 2 are not adjacent to the same vertex in Γ. Then it can be shown (see [HM06, Remark 7.16]) that there is a contracted edge of Γ, which leads to the image $\phi(\Gamma)$ being reducible. See Figure 9 for an example with no marked points. This means it is the union of two tropical rational curves C_A and C_B, of degrees d_A and d_B respectively, where 1 and 2 live on C_A, 1 lives on the line $x = (p_1)_1$, 2 lives on the line $y = (p_1)_2$, 3 and 4 live on C_B, $3d_A - 1$ of $\{5, ;n\}$ live on C_A, and $\phi(i) = p_{i-1}$ for $3 \leq i \leq n$. There are $\binom{3d-4}{3d_A-1}$ choices of the $3d_A - 1$ points that live on C_A. By Bézout the tropical curve C_A intersects the line $x = (p_1)_1$ (which is itself a tropical line) d_A times, and similarly for the line $y = (p_1)_2$. There are thus d_A choices for the image of 1, and d_A choices for the image of 2. Finally, the curves

C_1 and C_2 intersect in $d_A d_B$ points, so there are that many choices for the location of the contracted edge of Γ. This gives a total of

$$d_A^3 d_B \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B}$$

choices for such ϕ, so

$$M_d = N_d + d_A^3 d_B \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B}. \tag{2}$$

Alternatively, choose a phylogenetic tree $\overline{\Gamma}$ with four labelled leaves that looks like the one on the right of Figure 8, where the length b of the bounded edge is large. In this case we cannot have $\phi(1) = \phi(2)$, as that would mean that two other $\phi(i)$ would have to coincide, so two of the p_i would have to coincide, which is ruled out by the tropical general position hypothesis. So we are the second case above, where the image $\phi(\Gamma)$ is a reducible tropical rational curve. In this case we have 1 lying on the curve C_A of degree d_A, and 2 lying on the curve C_B of degree d_B. The point 3 lives on C_A and the point 4 lives on C_B. There are $3d_A - 2$ of the points $\{5, \dots, n\}$ living on C_A. There are thus $\binom{3d-4}{3d_A-2}$ choices for these points, d_A choices for the image of 1, d_B choices for the image of 2, and $d_A d_B$ choices for the image of the contracted edge of Γ. This gives

$$M_d = d_A^2 d_B^2 \binom{3d-4}{3d_A-2} N_{d_A} N_{d_B}. \tag{3}$$

Combining Equations 2 and 3 we obtain equation 1, and thus have the outline of the tropical proof of Kontsevich's formula.

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