AARMS TROPICAL GEOMETRY - LECTURES 13–15

DIANE MACLAGAN

For the rest of the week we consider the closure of $X \subset T^n$ in a toric variety. We start with taking the closure of X in $\mathbb{A}^n$. We will consider only the case that $\Bbbk \to K$, such as $\mathbb{C} \to \mathbb{C}\{\{t\}\}$.

Recall that $T^n \subset \mathbb{A}^n$, since $T^n = \{(x_1, \dots, x_n) : x_i \in \Bbbk, x_i \neq 0\}$, and $\mathbb{A}^n = \{(x_1, \dots, x_n) : x_i \in \Bbbk\}$. Given $X \subset T^n$, let $\overline{X}$ be the closure of X in $\mathbb{A}^n$. This is the smallest closed set in $\mathbb{A}^n$ containing X, so $\overline{X} = V(I)$ for some I and is as small as possible. Recall that if $V(I) \subsetneq V(J) \subset T^n$, then $\sqrt{J} \subsetneq \sqrt{I}$. If U is a subset of $\mathbb{A}^n$ then $\overline{U} = \bigcap_{U \subseteq V(I) : I \subseteq \Bbbk[x_1,\dots,x_n]} V(I)$.

Question: Given $X \subset T^n$ is $0 \in \overline{X}$?

The answer is given by the following theorem, which was first observed by Tevelev [Tev07].

Theorem 1. *Let $X \subset T^n_{\Bbbk}$, and let $\overline{X}$ be the closure of X in $\mathbb{A}^n$. Then $0 \in \overline{X}$ if and only if* $\operatorname{trop}(X) \cap \mathbb{R}^n_{>0} \neq \emptyset$*, where $\mathbb{R}^n_{>0} = \{(x_1, \dots, x_n) : x_i > 0$ for $1 \leq i \leq n\}$.*

Example: Let $X = V(x+y+1) \subset T^2$. Then $X = \{(a, -1-a) : a \in \Bbbk^*, a \neq -1\}$, so $\overline{X} = \{(a, -1-a) : a \in \Bbbk\} = X \cup \{(0,-1), (-1,0)\} = V(x+y+1) \subset \mathbb{A}^2$. The tropical variety $\operatorname{trop}(X)$ is shown in Figure 1. Note that $0 \notin \overline{X}$, and $\operatorname{trop}(X) \cap \mathbb{R}^2_{>0} = \emptyset$.

Example: Let $X = V(x^2 - y) \subset T^2$. Then $X = \{(a, a^2) : a \in \Bbbk^*\}$, and $\overline{X} = \{(a, a^2) : a \in K\} = X \cup \{(0,0)\} = V(x^2 - y) \subset \mathbb{A}^2$. Then $\operatorname{trop}(X)$ is the line $w_2 = 2w_1$, which contains the point $(1,2) \in \mathbb{R}^2_{>0}$, and $0 \in \overline{X}$.

Example: Let $X = V(xy-1) = \{(a, 1/a) : a \in \Bbbk^*\} \subset T^2$. Then $\overline{X} = X$, so $0 \notin \overline{X}$. The tropical variety is the line $w_1 + w_2 = 0$, which does not intersect the positive orthant.

We first note the following description of the ideal of $\overline{X} \subset \mathbb{A}^n$.

Lemma 2. *If $X = V(I) \subset T^n$ for $I \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then $\overline{X} = V(\overline{I}) \subset \mathbb{A}^n$, where $\overline{I} = I \cap \Bbbk[x_1, \dots, x_n]$.*

Proof. Let $\overline{X} = V(J)$ for $J \subset \Bbbk[x_1, \dots, x_n]$. Then since $X \subset \overline{X}$, we have $f(x) = 0$ for all $x \in X$ and $f \in J$, so $f \in \sqrt{I}$ when f is regarded as an element of $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $J' = J\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $J' \subseteq \sqrt{I}$. Now $J \subseteq J' \cap \Bbbk[x_1, \dots, x_n] \subseteq \sqrt{I} \cap$

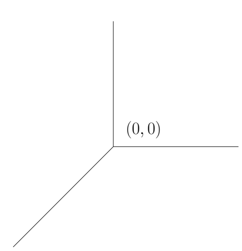

FIGURE 1.

$\Bbbk[x_1,\dots,x_n]$. We claim that $\sqrt{I}\cap\Bbbk[x_1,\dots,x_n]=\sqrt{\overline{I}}$. To see this, note that if $f\in\sqrt{I}\cap\Bbbk[x_1,\dots,x_n]$, then there is $N>0$ for which $f^N\in I$, and since $f\in\Bbbk[x_1,\dots,x_n]$ we have $f^N\in\overline{I}$, so $f\in\sqrt{\overline{I}}$. Conversely, if $f\in\sqrt{\overline{I}}$, then there is $N>0$ for which $f^N\in\overline{I}$, so $f^N\in I$, and $f\in\Bbbk[x_1,\dots,x_n]$, so $f\in\sqrt{I}\cap\Bbbk[x_1,\dots,x_n]$. Thus $J\subseteq\sqrt{\overline{I}}$, so $V(\overline{I})\subseteq V(J)=\overline{X}$. But if $x\in X$ then $f(x)=0$ for all $f\in\overline{I}$, so $X\subseteq V(\overline{I})$, and so $\overline{X}\subseteq V(\overline{I})$. Thus $\overline{X}=V(\overline{I})$. □

A key idea of the proof of Theorem 1 is the *inclusion of tori*. This will let us reduce to the case where X is a curve in T^2.

Definition 3. A morphism $\phi: T^n\to T^m$ is determined by a $\Bbbk$-algebra homomorphism $\phi^*:\Bbbk[y_1^{\pm1},\dots,y_m^{\pm1}]\to\Bbbk[x_1^{\pm1},\dots,x_n^{\pm1}]$. Note that the map ϕ^* goes in the reverse direction!

A point $a=(a_1,\dots,a_n)\in T^n$ corresponds to the maximal ideal

$$I_a=\langle x_1-a_1,\dots,x_n-a_n\rangle\in\Bbbk[x_1^{\pm1},\dots,x_n^{\pm1}].$$

Since the induced map $\Bbbk[y_1^{\pm1},\dots,y_m^{\pm1}]\to\Bbbk[x_1^{\pm1},\dots,x_n^{\pm1}]/I_a\cong\Bbbk$ is surjective, the kernel $\phi^{*-1}(I_a)$ is maximal, so is of the form $\langle y_1-b_1,\dots,y_m-b_m\rangle\in\Bbbk[y_1^{\pm1},\dots,y_m^{\pm1}]$ for some $b=(b_1,\dots,b_m)\in T^m$. We thus set $\phi(a)=b$.

Note that $\phi^*(y_i)$ is an invertible polynomial in $\Bbbk[x_1^{\pm1},\dots,x_n^{\pm1}]$ for $1\le i\le m$, so must be a monomial. We thus have

$$\phi^*(y_i)=x^{\mathbf{u}_i}\text{ for }\mathbf{u}_i\in\mathbb{Z}^n.$$

Thus a morphism $\phi: T^n\to T^m$ corresponds to a map $\psi:\mathbb{Z}^m\to\mathbb{Z}^n$ given by $\psi(\mathbf{e}_i)=\mathbf{u}_i$, where $\mathbf{e}_i$ is the ith standard basis vector of $\mathbb{R}^m$.

Exercise: The morphism ϕ is surjective if and only if ψ is injective. The morphism ϕ is injective if and only if ψ is surjective.

We record ψ by the $n\times m$ matrix U with columns $\mathbf{u}_1,\dots,\mathbf{u}_m$, so $\psi(\mathbf{v})=U\mathbf{v}$. The map $\phi: T^n\to T^m$ is given by $\phi(t_1,\dots,t_n)=(t^{\mathbf{u}_1},\dots,t^{\mathbf{u}_m})$, where $t^{\mathbf{u}_i}=t_1^{u_{i1}}t_2^{u_{i2}}\dots t_n^{u_{in}}$.

Given $X\subset T^n$ with $X=V(I)$, the closure $\overline{\phi(X)}$ of the image of X under ϕ is the variety $V(\phi^{*-1}(I))$.

Proposition 4. *Let $\phi: T^n\to T^m$ be a morphism of tori, with associated $n\times m$ matrix U. Let $X\subset T^n$ be a variety, and let $\overline{\phi(X)}$ be the closure of its image in T^m. Then $w\in\operatorname{trop}(X)$ if and only if $U^Tw\in\operatorname{trop}(\overline{\phi(X)})$.*

Proof. We denote by $X(K)$ the subvariety of T^n_K defined by the same equations as X. If $w\in\operatorname{trop}(X)$ then by the Fundamental Theorem there is $y\in X(K)$ with $\operatorname{val}(y)=w$. Then $\phi(y)=(y^{\mathbf{u}_1},\dots,y^{\mathbf{u}_m})\in\overline{\phi(X)}$, and $\operatorname{val}(\phi(y))=(\mathbf{u}_1\cdot\operatorname{val}(y),\dots,\mathbf{u}_m\cdot\operatorname{val}(y))=U^Tw$. So $w\in\operatorname{trop}(X)$ implies $U^Tw\in\operatorname{trop}(\overline{\phi(X)})$.

Conversely, if $\overline{w}\in\operatorname{trop}(\overline{\phi(X)})$, then there exists $\overline{y}\in\overline{\phi(X)}$ with $\operatorname{val}(\overline{y})=\overline{w}$. By [Pay07] the set of $\overline{y}\in\overline{\phi(X)}$ with $\operatorname{val}(\overline{y})=\overline{w}$ is Zariski dense in $\overline{\phi(X)}$, so we may assume that $\overline{y}\in\phi(X)$. Thus there is $y\in X$ with $\overline{y}=\phi(y)$. Then $\overline{w}=\operatorname{val}(\overline{y})=\operatorname{val}(\phi(y))=U^T\operatorname{val}(y)$, so there is $w=\operatorname{val}(y)\in\operatorname{trop}(X)$ with $\overline{w}=U^Tw$. □

We note that one direction of Proposition 4 can also be seen by arguments with initial ideals.

Lemma 5. *Let $I = \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $w \in \mathbb{R}^n$. Let $\phi : T^n \to T^m$ be given by $\phi(t)_i = t^{\mathbf{u}_i}$, and let U be the $n \times m$ matrix with columns the vectors $\mathbf{u}_i$. Let ϕ^* be the $\Bbbk$-algebra homomorphism $\Bbbk[y_1^{\pm 1}, \dots, y_m^{\pm 1}] \to \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then*

$$\mathrm{in}_{U^T w}(\phi^{*-1}(I)) \subseteq \phi^{*-1}(\mathrm{in}_w(I)).$$

Thus $w \in \mathrm{trop}(X)$ implies that $U^T w \in \mathrm{trop}(\overline{\phi(X)})$.

Proof. Let $f = \sum_{v \in \mathbb{N}^n} a_v y^v \in \phi^{-1}(I)$, so $\phi(f) = \sum_{v \in \mathbb{N}^n} a_v x^{Uv} \in I$. Then $\mathrm{in}_{U^T w} = \sum_{U^T w \cdot v = W} a_v y^v$, where $W = \min\{U^T w \cdot v : a_v \neq 0\} = \min\{v^T U^T w : a_v \neq 0\}$. However $\mathrm{in}_w(\phi(f)) = \sum_{w \cdot Uv = W'} a_v x^{Uv}$ where $W' = \min\{w \cdot Uv : a_v \neq 0\} = \min\{v^T U^T w : a_v \neq 0\} = W$. Thus $\{v : w \cdot Uv = W, a_v \neq 0\} = \{v : U^T w \cdot v, a_v \neq 0\}$, so $\mathrm{in}_w(\phi(f)) = \phi(\mathrm{in}_{U^T w}(f))$. Thus $\mathrm{in}_{U^T w}(f) \in \phi^{-1}(\mathrm{in}_w(I))$.

If $w \in \mathrm{trop}(X)$, then $\mathrm{in}_w(I) \neq \langle 1 \rangle$, so $\mathrm{in}_{U^T w}(\phi^{*-1}(I)) \neq \langle 1 \rangle$, and thus $U^T w \in \mathrm{trop}(\overline{\phi(X)})$. □

We next outline how to reduce the proof of Theorem 1 to the case where X is a curve.

Lemma 6. *Let $X \subset T^n$ with ideal $I \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\phi : T^m \to T^n$ be a morphism of tori with associated $m \times n$ matrix U and $\Bbbk$-algebra homomorphism $\phi^* : \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \Bbbk[y_1^{\pm 1}, \dots, y_m^{\pm 1}]$. Then $Y = \phi^{-1}(X) = V(\phi(I)) \subseteq T^m$.*

We have the containment of sets $U^T \mathrm{trop}(Y) \subset \mathrm{trop}(X) \cap \mathrm{trop}(\mathrm{im}(\phi))$.

Proof. Note that if $f = \sum_{v \in \mathbb{Z}^n} a_v x^v$, then $\phi^*(f) = \sum_{v \in \mathbb{Z}^n} a_v x^{Uv}$, so $\phi^*(f)(y) = f(\phi(y))$ for all $y \in T^m$. Let $y \in Y$. Then $\phi^*(f)(y) = f(\phi(y)) = 0$ for all $f \in I$, so $y \in V(\phi^*(I))$. Conversely, if $g(y) = 0$ for all $g \in \phi^*(I)$, then $\phi^*(f)(y) = 0$ for all $f \in I$, so $f(\phi(y)) = 0$ for all $f \in I$, and thus $\phi(y) \in V(I) = X$, so $y \in Y$.

Let $w \in \mathrm{trop}(Y)$. Then there is $y \in Y$ with $\mathrm{val}(y) = w$. Thus $\phi(y) \in X$, and so $\mathrm{val}(\phi(y)) = U^T w \in \mathrm{trop}(X)$. Since $\mathrm{trop}(\mathrm{im}(\phi)) = \mathrm{im}\, U^T$, it follows that $U^T w \in \mathrm{trop}(X) \cap \mathrm{trop}(\mathrm{im}(\phi))$. □

The morphism $\phi : T^n \to T^m$ extends to a morphism $\overline{\phi} : \mathbb{A}^n \to \mathbb{A}^m$ if and only if every entry of U is nonnegative. To reduce to the case that X is a curve, one intersects with a codimension $\dim(X) - 1$ subtorus of T^n that intersects X transversely in a curve $Y \subset T^m$ for $m = n - \dim(X) + 1$. We assume that the inclusion $\phi : T^m \to T^m$ has matrix U with positive entries. It then follows from Lemma 6 that if $0 \in \overline{X}$ then $0 \in \overline{Y}$, so given the curve case of Theorem 1 we know that there is $w \in \mathrm{trop}(Y)$ with $w \in \mathbb{R}^m_{>0}$. Then $U^T w \in \mathrm{trop}(X) \cap \mathbb{R}^m_{>0}$.

We will only prove Theorem 1 in the case where X is a curve in T^2. The elementary approach to a proof proposed in class does not work.

Proposition 7. *Let $X \subset T^2_{\Bbbk}$ be a curve, so $X = V(f)$ for some $f \in \Bbbk[x^{\pm 1}, y^{\pm 1}]$. Then $0 \in \overline{X}$ if and only if $\mathrm{trop}(X) \cap \mathbb{R}^2_{>0} \neq \emptyset$.*

Proof. Write $f = \sum_{(i,j) \in \mathbb{Z}^2} a_{ij} x^i y^j$. Let $I = \langle f \rangle \subseteq \Bbbk[x^{\pm 1}, y^{\pm 1}]$, and let $\overline{I} = I \cap \Bbbk[x, y]$. Then $\overline{I} = \langle f' \rangle$ for $f' = x^{-k} y^{-l} f$, where $k = \min\{i : a_{ij} \neq 0\}$, and $l = \min\{j : a_{ij} \neq 0\}$.

Now $0 \in \overline{X}$ if and only if $f'(0,0) = 0$, which occurs if and only if $a_{kl} \neq 0$. If $a_{kl} \neq 0$, then for all $w \in \mathbb{R}^2_{>0}$, we have $\mathrm{in}_w(f') = a_{kl}$, so $\mathrm{in}_w(I) = \langle 1 \rangle$, and thus

$\operatorname{trop}(X) \cap \mathbb{R}^2_{>0} = \emptyset$. Conversely, if $a_{kl} = 0$, then let $P = \operatorname{conv}((i-k, j-l) : a_{ij} \neq 0)$. By construction P lives in the positive orthant, and has a vertex v_1 of the form $(0, r)$ and a vertex v_∞ of the form $(s, 0)$. Let v_2 be the next vertex in counter-clockwise order from v_1. Let $w = (w_1, w_2)$ be the inner facet normal of the edge joining v_1 and v_2 (so $w \cdot u \geq w \cdot v_1 = w \cdot v_2$ for all $u \in P$). Note that $w_1, w_2 > 0$. Then $\operatorname{in}_w(f')$ is not a monomial, since its support contains the monomials with exponents giving rise to v_1 and v_2, and so $\operatorname{in}_w(f)$ is not a monomial, and thus $w \in \operatorname{trop}(X)$. □

Theorem 1 generalizes to the following theorem.

Theorem 8. *Let $X \subset T^n$ and let $\overline{X}$ be the closure of X in $\mathbb{A}^n$. Then for $\sigma \in \{1, \dots, n\}$*

$$\overline{X} \cap \{x \in \mathbb{A}^n : x_i = 0 \textit{ for all } i \in \sigma, x_i \neq 0 \textit{ for all } i \notin \sigma\} \neq \emptyset$$

if and only if there is $w \in \operatorname{trop}(X)$ with $w_i > 0$ for all $i \in \sigma$, and $w_i = 0$ for all $i \notin \sigma$.

The case $\sigma = \{1, \dots, n\}$ is Theorem 1. The condition on $w \in \operatorname{trop}(X)$ can be rephrased as asking that w lies in the relative interior of $\operatorname{pos}(\mathbf{e}_i : i \in \sigma)$. Examples of Theorem 8 can be seen by considering the subvarieties of T^2 at the start of the lecture.

One can also ask the same question about the closure of $X \subset T^n$ in $\mathbb{P}^n$. Recall that T^n embeds into $\mathbb{P}^n$ by the map $(x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$. Given $X \subset T^n$, let $\overline{X}$ now denote the closure of X in $\mathbb{P}^n$. This is the smallest projective variety containing X.

Example: Let $X = V(x_1 + x_2 + 1) \subset T^2$. Let $\overline{X}$ be the closure of X in $\mathbb{P}^2$ under the embedding $T^2 \to \mathbb{P}^2$ given by $(t_1, t_2) \mapsto (1 : t_2 : t_2)$. Then $\overline{X} = X \cup \{(1 : 0 : -1), (1 : -1 : 0), (0 : 1 : -1)\} = V(x_1 + x_2 + x_0)$. Note that $\overline{X} \cap \{x_i = 0\} \neq \emptyset$ for $i = 0, 1, 2$, while $\overline{X} \cap \{x_i = x_j = 0\} = \emptyset$ for all choices of $0 \leq i < j \leq 2$.

Note that the torus T^n is a group, with multiplication coordinatewise, and identity the element $(1, 1, \dots, 1) \in T^n$. The torus T^n acts on $\mathbb{P}^n$ by

$$(t_1, \dots, t_n) \cdot (x_0 : x_1 : \dots : x_n) = (x_0 : t_1 x_1 : t_2 x_2 : \dots : t_n x_n).$$

The orbits of T^n on $\mathbb{P}^n$ are indexed by proper subsets of $\{0, 1, \dots, n\}$ indicating which coordinates are zero.

Example: The torus T^2 acts on $\mathbb{P}^2$ by $(t_1, t_2) \cdot (x_0 : x_1 : x_2) = (x_0 : t_1 x_1 : t_2 x_2)$. The orbits of T^2 on $\mathbb{P}^2$ are:

$$\{T^2, \{(0 : 1 : t_2) : t_2 \neq 0\}, \{(1 : 0 : t_2) : t_2 \neq 0\}, \{(1 : t_1 : 0) : t_1 \neq 0\},$$
$$\{(1 : 0 : 0)\}, \{(0 : 1 : 0)\}, \{(0 : 0 : 1)\}.$$

These can be labelled by following subsets of $\{0, 1, 2\}$:

$$\{\emptyset, \{0\}, \{1\}, \{2\}, \{1, 2\}, \{0, 2\}, \{0, 1\}\}.$$

We denote by O_σ the orbit of $\mathbb{P}^n$ indexed by $\sigma \subset \{0, 1, \dots, n\}$.

Question: For $X \subset T^n$, let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Given $\sigma \subsetneq \{0, 1, \dots, n\}$, does $\overline{X}$ intersect O_σ?.

As before, then answer depends on the configuration of $\operatorname{trop}(X) \subset \mathbb{R}^n$. We will reduce this calculation to one in $\mathbb{A}^{n+1}$ that uses Theorem 8, by using the notion of the affine cone of X.

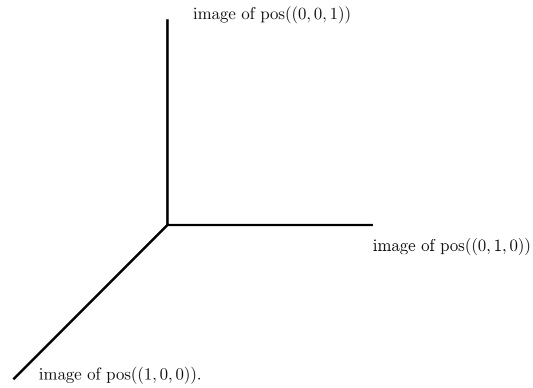

FIGURE 2.

Let $\tilde{T}^n = \{(t_0, t_1, \dots, t_n) : t_i \in \Bbbk^*\}$. Note that we have the short exact sequence

$$1 \to \Bbbk^* \to \tilde{T}^n \xrightarrow{\pi} T^n \to 1,$$

where 1 is the trivial group (written multiplicatively). The map $\Bbbk^* \to \tilde{T}^n$ is given by $t \mapsto (t, t, \dots, t)$, and the map $\pi : (t_0, \dots, t_n) \mapsto (t_1/t_0, \dots, t_n/t_0)$. This short exact sequence tropicalizes to

$$0 \to \operatorname{span}(\mathbf{1}) \to \mathbb{R}^{n+1} \xrightarrow{\operatorname{trop}(\pi)} \mathbb{R}^n \to 0,$$

where $\mathbf{1}$ is the vector $(1, 1, \dots, 1) \in \mathbb{R}^{n+1}$, the first map is the inclusion, and and $\operatorname{trop}(\pi) : (w_0, \dots, w_n) \to (w_1 - w_0, \dots, w_n - w_0)$.

Given $X \subset T^n$, the *affine cone* over X is $\tilde{X} = \pi^{-1}(X)$. The map $\pi^* : \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \Bbbk[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$ is given by $\pi^*(x_i) = x_i/x_0$ for $1 \leq i \leq n$. If $X = V(I) \subset T^n$, then $\tilde{X} = V(\pi^*(I)) \subset \tilde{T}^n$. Note that $\mathbf{1}$ lies in the lineality space of $\operatorname{trop}(\tilde{X})$, and that $\operatorname{trop}(X) = \operatorname{trop}(\tilde{X})/\mathbf{1}$.

Example: Let $X = V(x_1 + x_2 + 1) \subset T^2$. Then $\tilde{X} = V(x_1/x_0 + x_2/x_0 + 1) = V(x_1 + x_2 + x_0) \subset \tilde{T}^2$.

Let $\overline{\tilde{X}}$ be the closure of $\tilde{X}$ in $\mathbb{A}^{n+1}$. Recall that $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\Bbbk^*$. Then $\overline{X} = (\overline{\tilde{X}}) \setminus 0)/\Bbbk^*$. Thus $\overline{X}$ intersects the T^n-orbit indexed by $\sigma \subsetneq \{0, \dots, n\}$ if and only if the preimage $\operatorname{trop}(\tilde{X})$ of $\operatorname{trop}(X)$ intersects the relative interior of the appropriate $\operatorname{pos}(\mathbf{e}_i : i \in \sigma)$. We can also consider these sets in $\mathbb{R}^n$. Note that the image $\overline{\mathbf{e}}_0$ of $\mathbf{e}_0$ in $\mathbb{R}^n$ is $-\sum_{i=1}^n \overline{\mathbf{e}}_i$. Then $\overline{X}$ intersects the T^n-orbit indexed by σ if and only if $\operatorname{trop}(X) \cap \operatorname{relint}(\operatorname{pos}(\overline{\mathbf{e}}_i : i \in \sigma)) \neq \emptyset$.

Example: Figure 2 shows the fan in $\mathbb{R}^2$ whose cones are the sets $\operatorname{pos}(\overline{\mathbf{e}}_i : i \in \sigma)$ for $\sigma \subsetneq \{0, 1, 2\}$. Thus $\overline{X} \cap O_\sigma \neq \emptyset$ if and only if $\operatorname{trop}(X)$ intersects the relative interior of the appropriate cone. For an example, consider the variety $X = V(x_1 + x_2 + 1) \subset T^2$ analyzed above.

Note that T^n also acts on $\mathbb{A}^n$ by $(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1 x_1, \dots, t_n x_n)$, and Theorem 8 gave conditions for the closure $\overline{X}$ in $\mathbb{A}^n$ of a variety $X \subset T^n$ to intersect each orbit. The sets $\mathbb{A}^n$ and $\mathbb{P}^n$ are examples of *toric varieties*. These are varieties that contain a dense copy of T^n, and have an action of T^n on them extending the action of T^n on itself. They are (up to the technical notion of normalization) described by a polyhedral fan $\Sigma \subset \mathbb{R}^n$, the cones of which index T^n-orbits. If $\overline{X}$ is the closure of $X \subset T^n$ in a toric variety with fan Σ, then $\overline{X}$ intersects the T^n-orbit indexed by the cone $\sigma \in \Sigma$ if and only if $\mathrm{trop}(X)$ intersects the relative interior of σ.

The following proposition was also mentioned in class, and is used in the proof of the general case of the Fundamental Theorem.

Proposition 9. *Let $X \subset T^n$ be an irreducible variety of dimension d. Then for most choices of projection $\phi : T^n \to T^{d+1}$. the image $\overline{\phi(X)}$ is a hypersurface in T^{d+1}.*

Here by "most" we mean for a Zariski-open set of choices for the matrix U describing ϕ.

Proof. We first note that since X is irreducible, $\overline{\phi(X)}$ is irreducible of dimension at most d for any choice of projection ϕ. To see irreducibility, note that if $\overline{\phi(X)} = Y_1 \cup Y_2$ for $Y_1, Y_2 \subsetneq \overline{\phi(X)}$, then $X = X_1 \cup X_2$ with $X_i = \phi^{-1}(Y_i) \cap X$ for $i = 1, 2$. Then by the irreducibility of X without loss of generality we have $X = X_1$, so $X \subseteq \phi^{-1}(Y_1)$, and thus $\phi(X) \subseteq Y_1$, contradicting $Y_1 \subsetneq \overline{\phi(X)}$.

Since X is irreducible of dimension d and the generators of I have coefficients in $\Bbbk$, $\mathrm{trop}(X)$ is a pure d-dimensional fan in $\mathbb{R}^n$. Choose a d-dimensional cone $\sigma \in \mathrm{trop}(X)$, and choose an $n \times (d + 1)$ rank $d + 1$ matrix U with $\ker(U) \cap \mathrm{span}(\sigma) = \mathbf{0}$. Then $\{U^T w : w \in \sigma\}$ is a d-dimensional cone in $\mathbb{R}^{d+1}$, so $\mathrm{trop}(\overline{\phi(X)})$ has dimension at least d, and thus $\overline{\phi(X)}$ has dimension at least d. Since X has dimension d, $\overline{\phi(X)}$ has dimension at most d, and thus is a hypersurface in T^{d+1}. $\square$

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