# AARMS TROPICAL GEOMETRY - LECTURE 9 

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In this lecture we discuss another property of tropical varieties: that they are weighted balanced polyhedral complexes.
Example: Consider the vertex $(0,0)$ of $\operatorname{trop}(V(x+y))$. There are three rays leaving $(0,0): \operatorname{pos}((1,0)), \operatorname{pos}((0,1)), \operatorname{pos}((-1,-1))$. Note that we have

$$
\binom{1}{0}+\binom{0}{1}+\binom{-1}{-1}=\binom{0}{0} .
$$

Example: Let $X=V\left(t x^{2}+x+y+x y+t\right) \subset T_{K}^{2}$ for $K=\mathbb{C}\{\{t\}\}$. Then $\operatorname{trop}(X)$ is shown in Figure 1 .

Then the star of the vertex $(1,1)$ has rays spanned by $(1,0),(0,1)$, and $(-1-1)$, which add to $(0,0)$. This is also the star of the vertex $(-1,0)$. At the vertex $(0,0)$ the star has rays $(1,1),(-1,0)$, and $(0,-1)$, which add to $(0,0)$.
Example: Let $X=V\left(x^{2}+x y+t y+1\right) \subset T_{K}^{2}$ for $K=\mathbb{C}\{\{t\}\}$. Then $\operatorname{trop}(X)$ is shown in Figure 2.

The star of the vertex $(1,-1)$ has rays spanned by $(1,0),(0,-1)$, and $(-1,1)$, which add to zero. For the vertex $(0,0)$ the star has rays $(1,-1),(-1,-1)$, and $(0,1)$. In this case

$$
\binom{1}{-1}+\binom{-1}{-1}+\binom{0}{1}=\binom{0}{-1} \neq\binom{ 0}{0} .
$$

However,

$$
\begin{equation*}
\binom{1}{-1}+\binom{-1}{-1}+2\binom{0}{1}=\binom{0}{0} . \tag{1}
\end{equation*}
$$

We will define a notion of multiplicity on the top-dimensional polyhedra in $\operatorname{trop}(X)$ so that the corresponding sum is the zero vector. We review some commutative


Figure 1.


Figure 2.


Figure 3.
algebra to give the precise definition, and then give the intuitive definition as given in class.
Definition 1. Let $S=K\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$. The localization of $S$ at a prime ideal $P$ is the ring with elements $\{f / g: f, g \in S, g \notin P\}$. Given an ideal $I \subset S$, the multiplicity of $P$ in $I$ is mult $(P, S / I)=\operatorname{dim}_{K}\left(S_{P} / S_{P} I\right)$. If $I$ is radical, this is one if $V(P)$ is an irreducible component of $V(I)$, and zero otherwise.
Definition 2. Let $X=V(I)$ be an irreducible variety of dimension $d$ for $I \subset$ $K\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$. Then $\operatorname{trop}(X)$ is the support of a pure $d$-dimensional polyhedral complex $\Sigma$. Let $\sigma$ be a $d$-dimension polyhedron in $\Sigma$, and fix $w$ in the relative interior of $\sigma$. The multiplicity of the polyhedron $\sigma$ is the sum

$$
m_{\sigma}=\sum_{P \subset S} \operatorname{mult}\left(P, S /\left(\operatorname{in}_{w}(I)\right)\right.
$$

where the sum is over all prime ideals $P$ of $S$. If $\mathrm{in}_{w}(I)$ is radical, then $m_{\sigma}$ is the number of irreducible components of $V\left(\mathrm{in}_{w}(I)\right)$.

Example: When $f=x^{2}+x y+t y+1$, the initial terms of $f$ corresponding to each one is shown in Figure 3 .

Then $V(y+1)$ is irreducible, so this cone has multiplicity one. We have $V(x y+y)=$ $V(x+1)$ is also irreducible, so this cone gets multiplicity one. The varieties $V(x y+1)$


Figure 4.
and $V\left(x^{2}+x y\right)=V(x+y)$ are also irreducible, but $V\left(x^{2}+1\right)=V(x+i) \cup V(x-i)$, so the multiplicity of this last cone is two. This justifies Equation 1 .

Definition 3. A weighted polyhedral complex is a polyhedral complex $\Sigma$ with a positive integer on each top-dimensional polyhedron in $\Sigma$.

Let $\Sigma$ be a pure $d$-dimensional weighted polyhedral complex, and let $\sigma$ be a $(d-1)$ dimensional polyhedron in $\Sigma$. Let $V$ be the subspace $\operatorname{aff}(\sigma)-w$ for $w$ in the relative interior of $\sigma$. Then $V$ lies in the lineality space of $\operatorname{star}_{\Sigma}(\sigma)$, so $\operatorname{star}_{\Sigma}(\sigma) / V$ is a fan in $\mathbb{R}^{n} / V$. Let $\mathbf{u}_{\tau} \in \mathbb{R}^{n} / V$ be the image of the cone $\bar{\tau} \in \operatorname{star}_{\Sigma}(\sigma)$ in $\mathbb{R}^{n} / V$ for a $d$-dimensional polyhedron $\tau \in \Sigma$ with $\sigma$ a face of $\tau$. The polyhedral complex $\Sigma$ is balanced at $\sigma$ if

$$
\sum_{\tau} w_{\tau} \mathbf{u}_{\tau}=0
$$

where $w_{\tau}$ is the multiplicity of the polyhedron $\tau$.
The complex $\Sigma$ is balanced if it is balanced at every ( $d-1$ )-dimensional polyhedron.
Example: Let $X=V(x+y+z+1) \subset T_{K}^{3}$ for $K=\mathbb{C}\{\{t\}\}$. Then $\operatorname{trop}(X)$ is a two-dimensional fan with rays $\{(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)\}$ and cones spanned by any two of these. For $\sigma$ the ray through $(1,0,0)$ the fan $\operatorname{star}_{\operatorname{trop}(X)}(\sigma)$ is the one-dimensional fan in $\mathbb{R}^{3} / \operatorname{span}((1,0,0))$ with rays the images of $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$. If we identify $\mathbb{R}^{3} / \operatorname{span}((1,0,0))$ with $\mathbb{R}^{2}$ by projecting onto the last two coordinates of $\mathbb{R}^{3}$, these are the vectors $(1,0),(0,1)$, and $(-1,-1)$, which sum to zero, so $\operatorname{trop}(X)$ is balanced at $\sigma$.

Theorem 4. Let $X \subset T_{K}^{n}$ be an irreducible variety. Then $\operatorname{trop}(X)$ together with the multiplicity is a balanced weighted polyhedral complex.

Example: Let $f=x^{2}+y^{2}+x y^{2}+1$. Then $\operatorname{trop}(V(f))$ is shown in Figure 4 , together with the corresponding initial terms. Then

## Figure 5.

$$
\begin{aligned}
V\left(x^{+} 1\right) & =V(x+i) \cup V(x-i) \\
V\left(y^{2}+1\right) & =V(y+i) \cup V(y-i) \\
V\left(y^{2}+x y^{2}\right) & =V(x+1) \\
V\left(x^{2}+x y^{2}\right) & =V\left(x+y^{2}\right)
\end{aligned}
$$

Thus the multiplicity on the first two cones is two, and on the second two is one. This is shown in Figure 5. Since

$$
2\binom{0}{1}+2\binom{1}{0}+\binom{0}{-1}+\binom{-2}{-1}=\binom{0}{0}
$$

$\operatorname{trop}(X)$ is balanced.

