# AARMS TROPICAL GEOMETRY - LECTURE 8

DIANE MACLAGAN

The goal for today is to start describing the structure of the tropical variety.

**Example:** Let $f = x + y + 1 \in K[x^{\pm 1}, y^{\pm 1}]$. Then $V(f)$ is a line in $T^2$, and $\operatorname{trop}(V(f))$ is the standard "tropical line" we have seen multiple times, as shown in Figure 1

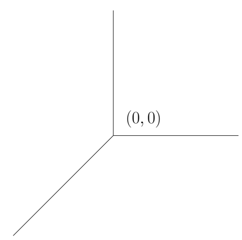


FIGURE 1.

**Example:** Let $f = x + y + z + 1 \in K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. Then $V(f)$ is a surface in $T^3$. We have $w \in \operatorname{trop}(V(f))$ if and only if

$$\begin{aligned} & w_1 = w_2 \leq w_3, 0 \\ \text{or } & w_1 = w_3 \leq w_2, 0 \\ \text{or } & w_1 = 0 \leq w_2, w_3 \\ \text{or } & w_2 = w_3 \leq w_1, 0 \\ \text{or } & w_2 = 0 \leq w_1, w_3 \\ \text{or } & w_3 = 0 \leq w_1, w_2. \end{aligned}$$

This is a fan with rays

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

The fan consists of all two-dimensional cones in $\mathbb{R}^3$ generated by any two of these rays. It intersects the sphere $S^3$ in the complete graph $K_4$.

**Example:** Let $\phi : T^d \to T^n$ be a subtorus embedded by

$$\phi : s = (s_1, \ldots, s_d) \mapsto (s^{a_1}, \ldots, s^{a_n}),$$

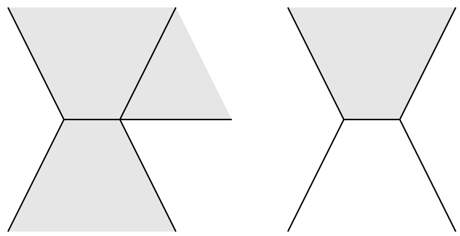

FIGURE 2. The complex on the left is pure, while the one on the right is not.

where $a_j \in \mathbb{Z}^d$ for $1 \leq i \leq n$, and $s^{a_j} = \prod_{i=1}^d s_i^{a_{ij}}$. We assume that the $d \times n$ matrix $A = (a_{ij})$ has rank $d$, so that $\phi$ is an embedding. Let $X = \text{im}(\phi) \cong T^d \subset T^n$. Then

$$\begin{aligned}
\text{trop}(X) &= \text{closure of } \{\text{val}(s^{a_1}), \ldots, \text{val}(s^{a_n}) : s = (s_1, \ldots, s_d) \in T_K^d\} \\
&= \text{closure of } \{\mathbf{a}_1 \cdot \text{val}(s), \ldots, \mathbf{a}_n \cdot \text{val}(s) : s \in T_K^s\} \\
&= \text{closure of } \{A^T \text{val}(s) : s \in T_K^d\} \\
&= \text{im } A^T.
\end{aligned}$$

So $\text{trop}(X)$ is a linear space of dimension $d$.

**Definition 1.** The *Minkowski sum* of two subsets $A, B \subset \mathbb{R}^n$ is the set

$$A + B = \{a + b : a \in A, b \in B\}.$$

**Definition 2.** The affine span of a polyhedron $P \subseteq \mathbb{R}^n$ is

$$\text{aff}(P) = v + \text{span}(u - v : u \in P)$$

where $v \in P$. Here the sum is an example of Minkowski addition. Note that this is independent of the choice of $v \in P$. The relative interior of $P$ is the interior of $P$ inside its affine span.

**Definition 3.** The dimension of a polyhedron $P$ is the dimension of its affine span. A polyhedral complex $\Sigma$ is *pure of dimension* $d$ if all maximal polyhedra in $\Sigma$ are $d$-dimensional.

**Note:** In each of the examples, $\text{trop}(X)$ is a pure polyhedral complex and $\dim(X) = \dim(\text{trop}(X))$. This is true in general.

Recall that the support of a polyhedral complex in $\mathbb{R}^n$ is the subset of $\mathbb{R}^n$ obtained by taking the union of all the polyhedra in the complex.

**Theorem 4.** *Let $X \subset T_K^n$ be an irreducible variety of dimension $d$ defined by the prime ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. There is a polyhedral complex $\Sigma$ that is pure of dimension $d$ whose support is* $\text{trop}(X)$.

The existence of the polyhedral complex we saw already in the discussion of the Gröbner complex last week. The new material here is that this complex is pure of dimension $d$.

To prove this we need some more Gröbner basics, which we will see first in an example.

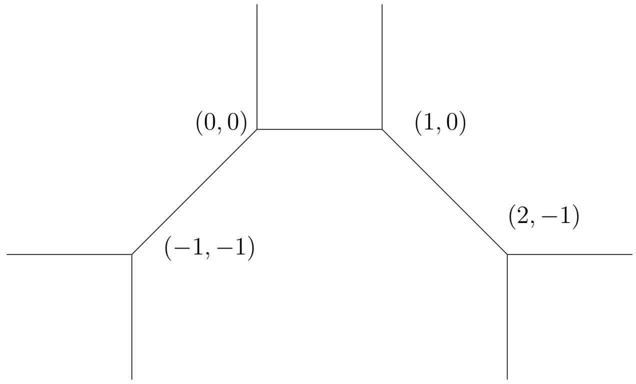


FIGURE 3.

**Example:** Let $f = tx^2y + x^2 + xy + t^2y + x + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Then $\mathrm{trop}(f) = \min(2x + y + 1, 2x, x + y, y + 2, x, 1)$, and $\mathrm{trop}(V(f))$ is shown in Figure 3.

For $w = (1, 0)$, we have $\mathrm{in}_w(f) = xy+x+1 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\mathrm{trop}(V(\mathrm{in}_w(f))) = \{v \in \mathbb{R}^2 : \langle \mathrm{in}_v(\mathrm{in}_w(f))\rangle \neq \langle 1\rangle\}$ is shown in Figure 4.

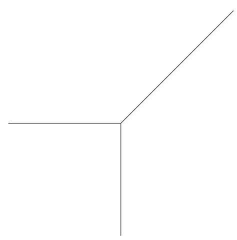

FIGURE 4.

For $w = (0, 0)$ we have $\mathrm{in}_w(f) = x^2 + xy + x$, and $\mathrm{trop}(\mathrm{in}_w(f)) = \{v \in \mathbb{R}^2 : \langle \mathrm{in}_v(\mathrm{in}_w(f))\rangle \neq \langle 1\rangle\}$ is shown in Figure 5.

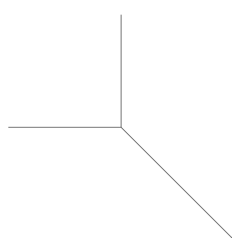

FIGURE 5.

If $w = (1/2, 0)$, then $\mathrm{in}_w(f) = x + xy$, and $\mathrm{trop}(V(\mathrm{in}_w(f)))$ is the $x$-axis $\{(x, y) \in \mathbb{R}^2 : y = 0\}$.

If $w = (1, 1)$, then $\mathrm{in}_w(f) = x + 1$, and $\mathrm{trop}(V(\mathrm{in}_w(f)))$ is the $y$-axis $\{(x, y) \in \mathbb{R}^2 : x = 0\}$.

Note that in all cases the set $\mathrm{trop}(V(\mathrm{in}_w(f)))$ looks like the piece of $\mathrm{trop}(V(f))$ "near" the polyhedron containing $w$. More formally, it is the *star*, which we now define, of the polyhedron containing $w$ in the polyhedral complex $\mathrm{trop}(V(f))$.

**Definition 5.** Let $\Sigma$ be a polyhedral complex, and let $\sigma \in \Sigma$ be a polyhedron. The *star* $\mathrm{star}_\Sigma(\sigma)$ of $\sigma \in \Sigma$ is a fan in $\mathbb{R}^n$ whose cones are indexed by those $\tau \in \Sigma$ for which $\sigma$ is a face of $\tau$. Fix $w \in \sigma$. Then the cone indexed by $\tau$ is the Minkowski sum

$$\bar{\tau} = \{v \in \mathbb{R}^n : \exists \epsilon > 0 \text{ with } w + \epsilon v \in \tau\} + \mathrm{aff}(\sigma) - w.$$

**Example:** For the polyhedral complex $\Sigma$ shown on the left of Figure 6, the affine span of the vertex $\sigma_1$ is just the vertex itself. The star is the standard line shown on the right. For $\sigma_2$ the affine span is the entire $y$-axis, and this is also the star.

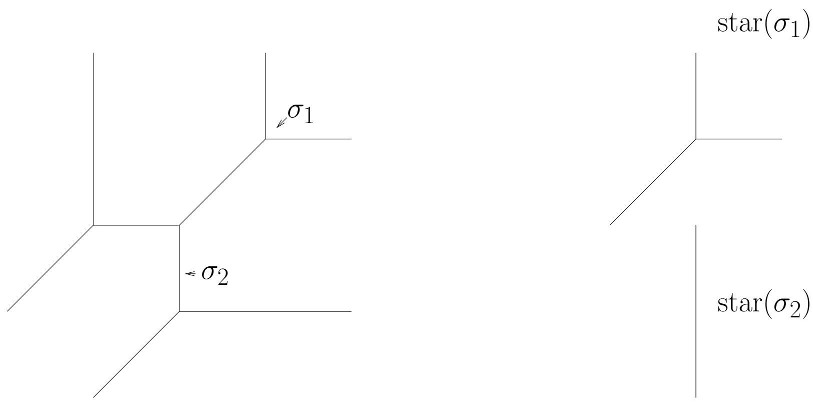


FIGURE 6.

**Lemma 6.** *Let $\Sigma$ be a polyhedral complex in $\mathbb{R}^n$, and $\sigma \in \Sigma$. Fix $w$ in the relative interior of $\sigma$. Then*

$$\mathrm{star}_\Sigma(\sigma) = \{v \in \mathbb{R}^n : w + \epsilon v \in \Sigma \text{ for sufficiently small } \epsilon > 0\}.$$

**Definition 7.** A subspace $V \subseteq \mathbb{R}^n$ is the *lineality space* of a polyhedron $P \subseteq \mathbb{R}^n$ if

$$x \in P \text{ implies } x + v \in P \text{ for all } v \in V.$$

If $V$ is the lineality space of a polyhedron $P$ then we often consider $P/V in \mathbb{R}^n/V$ for ease of visualization.

**Note:** The affine span $\mathrm{aff}(\sigma)$ of a polyhedron $\sigma \in \Sigma$ lies in the lineality space of every cone in the fan $\mathrm{star}(\sigma)$.

**Lemma 8.** *Let $I \subseteq K[x_1^{\pm 1} \dots, x_n^{\pm 1}]$, and fix $w, v \in \mathbb{R}^n$. Then there is $\epsilon > 0$ such that*

$$\mathrm{in}_v(\mathrm{in}_w(I)) = \mathrm{in}_{w+\epsilon v}(I).$$

*Proof.* We first note that it suffices to check that for all $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ there is $\epsilon > 0$ such that

$$\mathrm{in}_v(\mathrm{in}_w(f)) = \mathrm{in}_{w+\epsilon' v}(f)$$

for all $\epsilon' < \epsilon$.

To see this, note that $\mathrm{in}_v(\mathrm{in}_w(I))$ is finitely generated by $g_1, \dots, g_s \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and each generator $g_i$ is of the form $\mathrm{in}_v(\mathrm{in}_w(f_i))$ for some $f_i \in I$, so we can choose $\epsilon$ to be the minimum of the $\epsilon_i$ corresponding to these generating $f_i$. Then $g_i = \mathrm{in}_v(\mathrm{in}_w(f_i)) = \mathrm{in}_{w+\epsilon v}(f_i)$, so $\mathrm{in}_v(\mathrm{in}_w(I)) \subseteq \mathrm{in}_{w+\epsilon v}(I)$. Equality follows from the fact that we cannot have a proper inclusion of initial ideals.

We now prove the claim lemma for an individual polynomial. Let $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$. Then

$$\mathrm{in}_w(f) = \sum_{u \in \mathbb{Z}^n} \overline{c_u t^{w \cdot u - W}} x^u,$$

where $W = \min(\mathrm{val}(c_u) + w \cdot u : c_u \neq 0) = \mathrm{trop}(f)(w)$. Let $W' = \min(v \cdot u : \mathrm{val}(c_u) + w \cdot u = W)$. Then

$$\mathrm{in}_v(\mathrm{in}_w(f)) = \sum_{v \cdot u = W'} \overline{c_u t^{w \cdot u - W}} x^u.$$

Let $\delta = \min(\mathrm{val}(c_u) + w \cdot u - W : \mathrm{val}(c_u) + w \cdot u > w)$, and let $M = \max(v \cdot u : c_u \neq 0)$. Set $\epsilon = \delta/2M$, and $W'' = \min(\mathrm{val}(c_u) + (w + \epsilon v) \cdot u$. Then by construction we have

$$W'' = W + \epsilon W'$$

and

$$\{u : \mathrm{val}(c_u) + (w + \epsilon v) \cdot u = W''\} = \{u : \mathrm{val}(c_u) + w \cdot u = W, v \cdot u = W'\}.$$

Thus $\mathrm{in}_{w+\epsilon v}(f) = \mathrm{in}_v(\mathrm{in}_w(f))$. □

Recall that we say that $v \in \mathbb{R}^n$ is generic for $I$ if $\mathrm{in}_v(I)$ is a monomial ideal. The following corollary allows us to compute Gröbner bases with respect to nongeneric weight vectors using standard computer algebra packages.

**Corollary 9.** *Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $w \in \mathbb{R}^n$. Choose a vector $v \in \mathbb{R}^n$ that is generic for $\mathrm{in}_w(I)$. Then a Gröbner basis $\mathcal{G}$ for $I$ with respect to $w + \epsilon v$ for sufficiently small $\epsilon$ is a Gröbner basis for $I$ with respect to $w$, and $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g) : g \in \mathcal{G} \rangle$.*

*Proof.* Fix $\epsilon > 0$ such that $\mathrm{in}_{w+\epsilon v}(I) = \mathrm{in}_v(\mathrm{in}_w(I))$, the existence of which is guaranteed by Lemma 8. Let $\mathcal{G} = \{g_1, \dots, g_r\}$ be a Gröbner basis for $I$ with respect to $\mathrm{in}_{w+\epsilon v}$. Thus $\mathrm{in}_{w+\epsilon v}(I) = \langle \mathrm{in}_{w+\epsilon v}(g_1), \dots, \mathrm{in}_{w+\epsilon v}(g_r) \rangle$. The choice of $\epsilon$ was made to guarantee that $\mathrm{in}_{w+\epsilon v}(g_i) = \mathrm{in}_v(\mathrm{in}_v(g_i))$ for all $i$, so $\mathrm{in}_{w+\epsilon v}(I) = \langle \mathrm{in}_v(\mathrm{in}_w(g_1)), \dots, \mathrm{in}_v(\mathrm{in}_w(g_r)) \rangle = \mathrm{in}_v(\mathrm{in}_w(I))$, so $\{\mathrm{in}_w(g_1), \dots, \mathrm{in}_(g_r)\}$ is a Gröbner basis for $\mathrm{in}_w(I)$ with respect to $v$, and thus $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_r) \rangle$. □

**Corollary 10.** *Let $X \subset T_K^n$, with $X = V(I)$ for $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and let $\Sigma$ be a polyhedral complex whose support is $\mathrm{trop}(X) \subset \mathbb{R}^n$. Fix $w \in \mathrm{trop}(X)$, and let $\sigma$ be the polyhedron of $\Sigma$ containing $w$ in its relative interior. Then*

$$\mathrm{trop}(V(\mathrm{in}_w(I))) = \mathrm{star}_\Sigma(\sigma).$$

*Proof.* We have

$$\begin{aligned}
\mathrm{trop}(V(\mathrm{in}_w(I))) &= \{v \in \mathbb{R}^n : \mathrm{in}_v(\mathrm{in}_w(I)) \neq \langle 1 \rangle\} \\
&= \{v \in \mathbb{R}^n : \mathrm{in}_{w+\epsilon v}(I) \neq \langle 1 \rangle \text{ for sufficiently small } \epsilon > 0\} \\
&= \{v \in \mathbb{R}^n : w + \epsilon v \in \mathrm{trop}(X) \text{ for sufficiently small } \epsilon > 0\} \\
&= \mathrm{star}_\Sigma(\sigma),
\end{aligned}$$

where the last equality is by Lemma 6. □

To prove Theorem 4 we need the following Lemma.

**Lemma 11.** *Let $Y \subseteq T_{\Bbbk}^n$ be equidimensional of dimension $d$ (all irreducible components have the same dimension). Suppose that* $\operatorname{trop}(Y)$ *is a linear subspace of $\mathbb{R}^n$. Then there is a $d$-dimensional subtorus $T \subset T_{\Bbbk}^n$ such that $V(I)$ consists of finitely many $T$-orbits.*

*Proof.* For a proof, see Lemma 9.9 of [Stu02] □

*Proof of Theorem 4.* Let $\Sigma$ be a polyhedral complex with support $\operatorname{trop}(X)$, and let $\sigma$ be maximal polyhedron in $\Sigma$ (so $\sigma$ is not a proper face of any polyhedron in $\Sigma$). We need to show that $\dim(\sigma) = d$. Fix $w$ in the relative interior of $\sigma$. By Corollary 10 we have $\operatorname{trop}(\operatorname{in}_w(I)) = \operatorname{star}_\Sigma(\sigma)$. Since $\sigma$ is maximal, we have that $\operatorname{star}_\Sigma(\sigma) = \operatorname{aff}(\sigma)$ is a $\dim(\sigma)$-dimensional linear subspace. Since $I$ is prime, it follows from a resultof Kalkbrener and Sturmfels [KS95] that $V(\operatorname{in}_w(I))$ is equidimensional, so all irreducible components have the same dimension. By Lemma 11 it follows that there is a subtorus $T \subset T_K^n$ of dimension $\dim(\sigma)$ for which $V(\operatorname{in}_w(I))$ is the union of finitely many $T$-orbits. Since $\dim(V(\operatorname{in}_w(I))) = \dim(I) = d$ (Exercise!), it follows that $\dim(\sigma) = d$. □

## References

[KS95] Michael Kalkbrener and Bernd Sturmfels, *Initial complexes of prime ideals*, Adv. Math. **116** (1995), no. 2, 365–376. MR **1363769 (97g:**13043)

[Stu02] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2002. MR **1925796 (2003i:**13037)