AARMS TROPICAL GEOMETRY - LECTURE 5

DIANE MACLAGAN

The goal for today is to discuss Gröbner bases in our contexts. From now on we will always have K being an algebraically closed field with a nontrivial valuation (such as $\mathbb{C}\{\{t\}\}$) with residue field $\Bbbk$. We will discuss Gröbner bases in three different contexts.

The homogeneous case. We first consider the case where there is an inclusion of $\Bbbk$ into K with the image having valuation zero. This is the case for the Puiseux series, but not for all possible K. When an ideal has generators in $\Bbbk \subset K$, we say that the corresponding variety is defined over $\Bbbk$, and that we are in the *constant coefficients* case.

In this case, we first let $S = \Bbbk[x_0, \dots, x_n]$. Fix $w \in \mathbb{R}^n$. Given a polynomial $f = \sum_{u \in \mathbb{N}^n} c_u x^u \in S$, set $W = \min\{(0,w) \cdot u : c_u \neq 0\}$. Then

$$\mathrm{in}_w(f) = \sum_{(0,w)\cdot u = W} c_u x^u.$$

If I is a homogeneous ideal, then we set

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

Example: Let $f = x_0^2 + 3x_0x_1$. When $w = 2$, $\mathrm{in}_w(f) = x_0^2$. When $w = 0$, $\mathrm{in}_w(f) = x_0^2 + 3x_0x_1$. When $w = -3$, $\mathrm{in}_w(f) = 3x_0x_1$. If $I = \langle x_0^2 + 3x_0x_1 \rangle$, then $\mathrm{in}_2(I) = \langle x_0^2 \rangle$.
Example: $I = \langle x_0x_2 - x_1^2, x_0x_1 - x_2^2 \rangle$. Take $w = (2,3)$. Then $\mathrm{in}_w(x_0x_2 - x_1^2) = x_0x_2$, and $\mathrm{in}_w(x_0x_1 - x_2^2) = x_0x_1$. *However* $\mathrm{in}_w(I) \neq \langle x_0x_2, x_0x_1 \rangle$, since $x_1^3 - x_2^3 \in I$, and $\mathrm{in}_w(x_1^3 - x_2^3) = x_1^3$. In this case $\mathrm{in}_w(I) = \langle x_0x_2, x_0x_1, x_1^3 \rangle$.

Definition 1. A set $\{g_1, \dots, g_s\} \subset I$ is a Gröbner basis for w if $\mathrm{in}_w(I)$ is generated by $\{\mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_s)\}$.

Example: With I as above, and $w = (-1,-1)$, $\mathrm{in}_w(I) = \langle x_1^2, x_2^2 \rangle$. With $w = (0,0)$, $\mathrm{in}_w(I) = \langle x_0x_2 - x_1^2, x_0x_1 - x_2^2 \rangle$. With $w = (1,2)$, $\mathrm{in}_w(x_0x_2 - x_1^2) = x_0x_2 - x_1^2$, and $\mathrm{in}_w(x_0x_1 - x_2^2) = x_0x_1$, and we have $\mathrm{in}_w(I) = \langle x_0x_2 - x_1^2, x_0x_1 \rangle$.

Definition 2. An ideal in S is monomial if it is generated by monomials. We say that w is generic with respect to I if $\mathrm{in}_w(I)$ is a monomial ideal.

Lemma 3. *If w is generic then the monomials not in $\mathrm{in}_w(I)$ form a $\Bbbk$-basis for S/I.*

We put an equivalence relation on $\mathbb{R}^n$ by setting $w \sim w'$ if $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$.
Example: With I as above, for $w = (-2,-3)$, we have $\mathrm{in}_w(I) = \langle x_1^2, x_2^2 \rangle$, so $(-1,-1) \sim (-2,-3)$.

Theorem 4. *The set*

$$C[w] := \{w' \in \mathbb{R}^n : \mathrm{in}_w(I) = \mathrm{in}_{w'}(I)\}$$

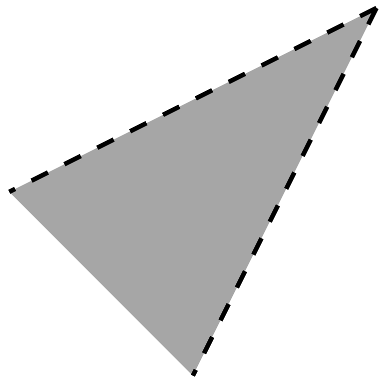

Figure 1.

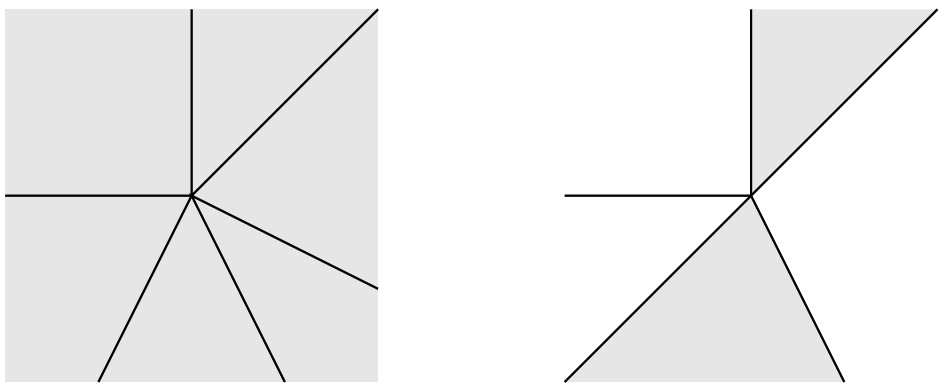

Figure 2. Polyhedral fans

is a relatively open polyhedral cone. This means that it can be described by equations and strict inequalities.

For a proof of this, see [Stu96, Chapter 2], or [Mac].

Example: Let I be as above, and $w = (-1, -1)$. Then

$$C[w] = \{w' \in \mathbb{R}^2 : 2w'_1 < w'_2, 2w'_1 < w'_2\}.$$

This is shown in Figure 1.

Definition 5. A polyhedral cone is the intersection of finitely many halfspaces with the corresponding hyperplanes passing through the origin. It is thus of the form

$$C = \{x : Ax \leq 0\}$$

where A is a $d \times n$ matrix. A hyperplane H in $\mathbb{R}^n$ is supporting for a cone C if C lies in one of the two halfspaces determined by H. A face of C is the intersection of C with a supporting hyperplane.

A fan is a collection of polyhedral cones, the intersection of any two of which is a face of each.

Theorem 6. *For a fixed ideal I the collection $\{\overline{C[w]} : w \in \mathbb{R}^n\}$ is a polyhedral fan.*

Definition 7. This fan is called the Gröbner fan of I.

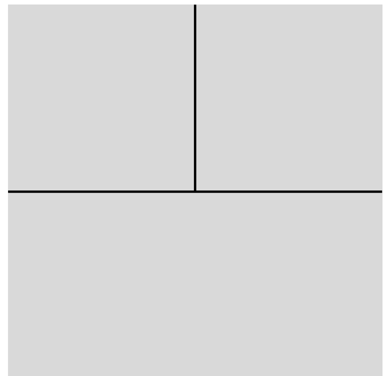

FIGURE 3. Not a polyhedral fan

Example: Let $I = \langle x_0x_1 - x_2^2, x_0x_2 - x_1^2 \rangle$. Then the Gröbner fan for I is shown in Figure 4, where the ideals corresponding to the cones are given in the following table.

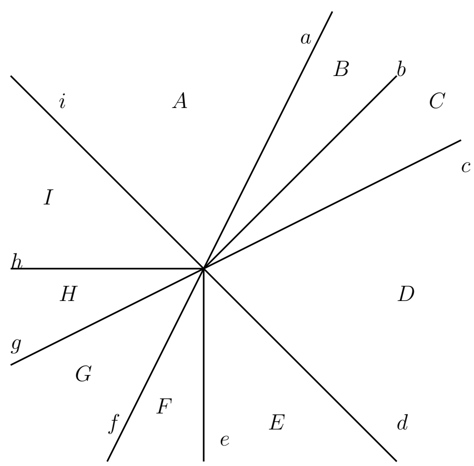

FIGURE 4.

Cone	Initial ideal	Cones	Initial ideal
A	$\langle x_0x_1, x_1^2, x_0^2x_2 \rangle$	a	$\langle x_0x_1, x_0x_2 - x_1^2 \rangle$
B	$\langle x_0x_2, x_0x_1, x_2^3 \rangle$	b	$\langle x_0x_2, x_0x_1, x_1^3 - x_2^3 \rangle$
C	$\langle x_0x_2, x_0x_1, x_1^3 \rangle$	c	$\langle x_0x_2, x_0x_1 - x_2^2 \rangle$
D	$\langle x_0x_2, x_2^2, x_0^2x_1 \rangle$	d	$\langle x_0x_2, x_2^2, x_0^2x_1 - x_1^2x_2 \rangle$
E	$\langle x_0x_2, x_2^2, x_2^2x_2, x_0^3x_1 \rangle$	e	$\langle x_0x_2, x_2^2, x_1^2x_2, x_0^3x_1 - x_1^4 \rangle$
F	$\langle x_0x_2, x_2^2, x_1^2x_2, x_1^4 \rangle$	f	$\langle x_0x_2 - x_1^2, x_2^2 \rangle$
G	$\langle x_1^2, x_2^2 \rangle$	g	$\langle x_0x_1 - x_2^2, x_1^2 \rangle$
H	$\langle x_0x_1, x_1^2, x_1x_2^2, x_2^4 \rangle$	h	$\langle x_0x_1, x_1^2, x_1x_2^2, x_0^3x_2 - x_2^4 \rangle$
I	$\langle x_0x_1, x_1^2, x_1x_2^2, x_0^3x_2 \rangle$	i	$\langle x_0x_1, x_1^2, x_0^2x_2 - x_1x_1^2 \rangle$

There is software, called `gfan` [Jen], written by Anders Jensen, that will compute the Gröbner fan of an ideal.

The torus case. We now consider the case when $S = \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Given a Laurent polynomial $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$, and $w \in \mathbb{R}^n$, we set $W = \min\{w \cdot u : c_u \neq 0\}$, and then

$$\mathrm{in}_w(f) = \sum_{w \cdot u = W} c_u x^u.$$

If I is an ideal in S, then the initial ideal is

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

We make the same caveats as before on the fact that the initial ideal is not necessarily generated by the initial terms of generators.
Example: Let $f = x + 1 \in \Bbbk[x^{\pm 1}]$, and let $I = \langle x + 1 \rangle$. When $w = 1$, we have $\mathrm{in}_w(f) = 1$, and $\mathrm{in}_w(I) = \langle 1 \rangle$. When $w = -1$, $\mathrm{in}_w(f) = x$ and $\mathrm{in}_w(I) = \langle x \rangle = \langle 1 \rangle$.
Example: Let $f = x + y + 1 \in \Bbbk[x^{\pm 1}, y^{\pm 1}]$, and let $I = \langle f \rangle$. For $w = (1, 1)$, $\mathrm{in}_w(f) = 1$, and $\mathrm{in}_w(I) = \langle 1 \rangle$. For $w = (1, 0)$, $\mathrm{in}_w(f) = y + 1$, and $\mathrm{in}_w(I) = y + 1$. If $w = (1, -1)$ then $\mathrm{in}_w(f) = y$, and $\mathrm{in}_w(I) = \langle y \rangle = \langle 1 \rangle$.

The Gröbner fan does not exist in the same fashion, as we can see from this example that $\{w : \mathrm{in}_w(I) = \langle 1 \rangle\}$ is not convex. However ignoring these cases gives a cone. The support of a polyhedral fan in $\mathbb{R}^n$ is the set of those $w \in \mathbb{R}^n$ lying in some cone of the fan. It follows from the following proposition that the set of w with $\mathrm{in}_w(I) \neq \langle 1 \rangle$ has the support of a polyhedral fan.

If $f = \sum c_u x^u \in \Bbbk[x_1, \dots, x_n]$, let $W = \max\{|u| : c_u \neq 0\}$, where $|u| = \sum_{i=1}^n u_i$. The homogenization, $\tilde{f}$ of f is $\tilde{f} = \sum c_u x_0^{W - |u|} x^u \in \Bbbk[x_0, \dots, x_n]$.

Proposition 8. *Let I be an ideal in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\bar{I} = I \cap \Bbbk[x_1, \dots, x_n]$, and let $J \subset \Bbbk[x_0, \dots, x_n] = \langle \tilde{f} : f \in \bar{I} \rangle$. Then*

$$\mathrm{in}_w(J)|_{x_0 = 1} = \mathrm{in}_w(I).$$

Proof. Exercise. □

Remark 9. The variety in $\mathbb{P}^n$ of the ideal J from Proposition 8 is the *projective closure* of the variety of I in T^n. This is the Zariski closure in $\mathbb{P}^n$ of the image of the variety of $I \subset T^n$ under the map $i : T^n \to \mathbb{P}^n$ given by $x \mapsto (1 : x)$.

Corollary 10. *Let $I \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and let J be the ideal defined in Proposition 8. Then the support of the subfan of the Gröbner fan of J consisting of those cones σ for which $(\mathrm{in}_w(J) : x_0^\infty) \neq \langle 1 \rangle$ for $w \in \sigma$ is $\{w \in \mathbb{R}^n : \mathrm{in}_w(I) \neq \langle 1 \rangle$.*

Proof. By Proposition 8, we have $\mathrm{in}_w(I) \neq \langle 1 \rangle$ if and only if there is no power of x_0 in $\mathrm{in}_w(J)$, and thus if and only if $(\mathrm{in}_w(J) : x_0^\infty) \neq \langle 1 \rangle$. □

Nonconstant coefficients. We now consider the case that $S = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Fix $w \in \mathbb{R}^n$, and let $f = \sum_{u \in \mathbb{Z}^n} c_u x_u \in S$. Let $W = \min\{\mathrm{val}(c_u) + w \cdot u : c_u \neq 0\}$. Then

$$\mathrm{in}_w(f) = \overline{t^{-W} \sum_{u \in \mathbb{Z}^n} c_u t^{w \cdot u} x^u} \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

and

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

Example: Let $f = (t + t^2)x + t^2y + t^4$, and $w = (0, 0)$. Then $W = 1$, and $\mathrm{in}_w(f) = \overline{(1 + t)x} = x$. When $w = (4, 2)$, $W = 4$, and $\mathrm{in}_w(f) = y + 1$. When $w = (2, 1)$, $W = 3$, and $\mathrm{in}_w(f) = x + y$.

Definition 11. A polyhedron is the intersection of finitely many (affine) halfspaces. Unlike a polyhedral cones the boundary (affine) hyperplanes are not required to pass through the origin. An affine hyperplane H is supporting for a polyhedron P if $P \cap H \neq \emptyset$ and P lies on one side of H. A face of P is the intersection of P with a supporting hyperplane, or P itself. A *polyhedral complex* in $\mathbb{R}^n$ is a collection of polyhedra in $\mathbb{R}^n$, the intersection of any two of which is a face of each.

As in the constant coefficient $\Bbbk$ case we can also consider initial ideal in the homogenized polynomial ring $K[x_0, \ldots, x_n]$. Instead of a Gröbner fan, though, there is now a *Gröbner complex.* Each $w \in \mathbb{R}^n$ determines a relatively open polyhedron in $\mathbb{R}^n$ on which the initial ideal is constant. Throwing away those polyhedra for which the corresponding initial ideal contains a power of x_0, we obtain the following theorem, whose proof we will omit.

Theorem 12. *The set of* $w \in \mathbb{R}^n$ *for which* $\mathrm{in}_w(I) \neq \langle 1 \rangle$ *is the support of a polyhedral complex.*

Example: Let $f = tx^2 + 2xy + 3ty^2 + 4x + 5y + 6t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$, and let $I = \langle f \rangle$ be the ideal generated by I. Then the set $\{w \in \mathbb{R}^2 : \langle \mathrm{in}_w(I) \neq \langle 1 \rangle\}$ is illustrated in Figure 5. The initial ideals corresponding to the labelled edges are as

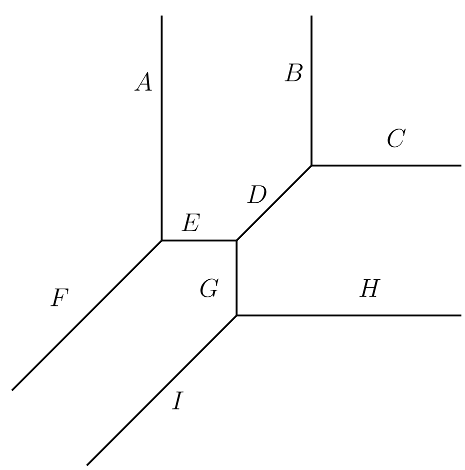

FIGURE 5.

listed in the following table.

Cone	Initial ideal
A	$\langle x^2 + 4x \rangle$
B	$\langle 4x + 6 \rangle$
C	$\langle 5y + 6 \rangle$
D	$\langle 4x + 5y \rangle$
E	$\langle 2xy + 4x \rangle$
F	$\langle x^2 + 2xy \rangle$
G	$\langle 2xy + 5y \rangle$
H	$\langle 3y^2 + 5y \rangle$
I	$\langle 2xy + 3y^2 \rangle$

References

[Jen] Anders N. Jensen, *Gfan, a software system for Gröbner fans*. Available at `http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html`.

[Mac] Diane and Thomas Rekha R. Maclagan, *Computational Algebra and Combinatorics of Toric Ideals*. Available at `http://www.warwick.ac.uk/staff/D.Maclagan/papers/indialectures.pdf.gz`.

[Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR **1363949 (97b:**13034)