AARMS TROPICAL GEOMETRY - LECTURE 4

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Today we will discuss valuations and Puiseux series.

Let K be a field. We denote by K^* the nonzero elements of K. A *valuation* on K is a function $\mathrm{val}\colon K \to \mathbb{R} \cup \infty$ satisfying

(1) $\mathrm{val}(a) = \infty$ if and only if $a = 0$,
(2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ and
(3) $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in K^*$.

We will always assume that $1 \in \mathrm{im}(\mathrm{val})$. Since $(\lambda\,\mathrm{val})\colon K \to \mathbb{R}$ is a valuation for any valuation val and $\lambda \in \mathbb{R}_{>0}$, this is not a serious restriction.

Example: $K = \Bbbk(x)$, the ring of rational functions. We can write any function $f/g \in K$ as a Laurent series $h = \sum h_i x^i$ where $h_i = 0$ for $i \ll 0$. Then $\mathrm{val}(f/g) = \min(i : h_i \neq 0)$. If i is the lowest exponent occuring in f and j is the lowest exponent occuring in g, then $\mathrm{val}(f/g) = i - j$.

Example: $K = \mathbb{Q}$, and $\mathrm{val}_p(q) = j$ when $q = p^j a/b$, where p does not divide a or b. For example $\mathrm{val}_2(12/5) = 2$, while $\mathrm{val}_2(1/10) = -1$. This the p-adic valuation.

Lemma 1. *If* $\mathrm{val}(a) \neq \mathrm{val}(b)$ *then* $\mathrm{val}(a+b) = \min(\mathrm{val}(a), \mathrm{val}(b))$.

Proof. Without loss of generality we may assume that $\mathrm{val}(b) > \mathrm{val}(a)$. Since $1^2 = 1$, we have $\mathrm{val}(1) = 0$, and so $(-1)^2 = 1$ implies $\mathrm{val}(-1) = 0$ as well. Thus $\mathrm{val}(-b) = \mathrm{val}(b)$, so $\mathrm{val}(a) \geq \min(\mathrm{val}(a+b), \mathrm{val}(-b)) = \min(\mathrm{val}(a+b), \mathrm{val}(b))$, and so $\mathrm{val}(a) \geq \mathrm{val}(a+b)$. But $\mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b)) = \mathrm{val}(a)$, and thus $\mathrm{val}(a+b) = \mathrm{val}(a)$. □

Given a valuation val we define the *valuation ring*

$$R = \{a \in K : \mathrm{val}(a) \geq 0\} \cup \{0\}.$$

This is closed under addition and multiplication, since $\mathrm{val}(a), \mathrm{val}(b) \geq 0$ implies $\mathrm{val}(ab), \mathrm{val}(a+b) \geq 0$. It has a unique maximal ideal

$$\mathfrak{m} = \{a \in K : \mathrm{val}(a) > 0\} \cup \{0\}.$$

To see that $\mathfrak{m}$ is the unique maximal ideal, it suffices to note that if $a \in R \setminus \mathfrak{m}$ then a is a unit in R. Indeed, if $a \in R \setminus \mathfrak{m}$, then $\mathrm{val}(a) = 0$, so $\mathrm{val}(a^{-1}) = -\mathrm{val}(a) = 0$, so $a^{-1} \in R$. The *residue field* is

$$\Bbbk = R/\mathfrak{m}.$$

Example: If $K = \Bbbk((x))$ is the quotient ring of $\Bbbk[[x]]$, then $R = \Bbbk[[x]]$, and $\mathfrak{m} = \Bbbk$.

Example: In the case that $K = \mathbb{Q}$ and val is the p-adic valuation, we have $R = \{p^j a/b : j \geq 0\} \cup \{0\}$. Exercise: Check that the residue field is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Example: Let $R_n = \Bbbk[[t^{1/n}]]$, and let $\Bbbk((t^{1/n}))$ be its quotient field. Let $K = \bigcup_{n \geq 1} \Bbbk((t^{1/n}))$, which we denote by $\Bbbk\{\{t\}\}$. Note that K is closed under addition and multiplication, and is thus a field. The field K is the ring of *Puiseux series*. An element of K has the form $\sum_{q\in\mathbb{Q}} a_q t^q$ where $\{q : a_q \neq 0\}$ is bounded below and has a common denominator.

The field $\Bbbk((t^{1/n}))$ has a valuation like that on the ring of rational functions. This induces a valuation $\mathrm{val} : K \to \mathbb{R}\cup\infty$. If $a = \sum_{q\in\mathbb{Q}} a_q t^q \in K$, then $\mathrm{val}(a) = \min\{q : a_q \neq 0\}$.

Puiseux series are useful because they are algebraically closed, as we now prove.

Theorem 2. *If $\Bbbk$ is an algebraically closed field of characteristic zero, then $K = \Bbbk\{\{t\}\}$ is algebraically closed.*

I learned the following proof from Thomas Markwig, and it is closely modelled on the one he gives in his paper [Mar07] on a generalization of the Puiseux series.

Proof. We need to show that given a polynomial $F = \sum_{i=0}^n c_i x^i \in S = K[x]$ there is $y \in K$ with $F(y) = \sum_{i=0}^n c_i y^i = 0$. In principle the idea is to build y up as a Puiseux series by successive powers of t.

We first note that we may assume the following properties of F:

(1) $\mathrm{val}(c_i) \geq 0$ for all i,
(2) There is some j with $\mathrm{val}(c_j) = 0$,
(3) $c_0 \neq 0$, and
(4) $\mathrm{val}(c_0) > 0$.

To see this, note that if $\alpha = \min\{\mathrm{val}(c_i) : 0 \leq i \leq n\}$ then multiplying F by $t^{-\alpha}$ does not change the existence of a root of F, which deals with the first two properties. If $c_0 = 0$ then $y = 0$ is a root so there is nothing to prove.

To make the last assumption, suppose that F satisfies the first three assumptions but $\mathrm{val}(c_0) = 0$. If $\mathrm{val}(c_n) > 0$ then we can form $G(x) = x^n F(1/x) = \sum_{i=0}^n c_{n-i} x^i$, which has the desired form, and if $G(y') = 0$ then $F(1/y') = 0$. If $\mathrm{val}(c_0) = \mathrm{val}(c_n) = 0$ then consider the polynomial $f := \overline{F} \in \Bbbk[x]$ that is the image of F in $K[x]/\mathfrak{m}K[x]$. This which is not constant since $\mathrm{val}(c_n) = 0$. Since $\Bbbk$ is algebraically closed, we can choose a root $\lambda \in \Bbbk$ of f. Then

$$F'(x) := F(x+\lambda) = \sum_{i=0}^n (\sum_{j=i}^n c_j \binom{j}{i} \lambda^{j-i}) x^i$$

has constant term $F'(0) = F(\lambda)$ with positive valuation, and F' still satisfies the first three properties. If y' is a root of F', then $y' + \lambda$ is a root of F.

Set $F_0 = F$. We will construct a sequence of polynomials $F_i = \sum_{j=0}^n c_j^i x^j$. Suppose, as we have shown we may assume for $i = 0$, that F_i satisfies conditions 1 to 4 above. The *Newton polygon* of F_i is the convex hull of the points $\{(i,j) :$ there is k with $k \leq i, \mathrm{val}(c_k) \leq j\} \subset \mathbb{R}^2$. There is an edge of the Newton polygon with negative slope connecting the vertex $(0, \mathrm{val}(c_0^i))$ to a vertex $(k_i, \mathrm{val}(c_{k_i}^i))$. Let

$$w_i = \frac{\mathrm{val}(c_0^i) - \mathrm{val}(c_{k_i}^i)}{k_i}.$$

Let f_i be the image in $\Bbbk[x]$ of the polynomial $t^{-\mathrm{val}(c_0^i)} F(t^{w_i} x) \in K[x]$. Note that f_i has degree k_i, and has nonzero constant term. Since $\Bbbk$ is algebraically closed we can find a root λ_i of f_i. Let r_{i+1} be the multiplicity of λ_i as a root of f_i, so $f_i = (x-\lambda_i)^{r_{i+1}} g_i(x)$, where $g_i(\lambda_i) \neq 0$. Set

$$F_{i+1}(x) = t^{-\operatorname{val}(c_0^i)} F_i(t^{w_i}(x+\lambda_i)) = \sum_{j=0}^{n} c_j^{i+1} x^j.$$

Note that the coefficients c_j^{i+1} are given by the formula

$$c_j^{i+1} = \sum_{l=j}^{n} c_l^i t^{lw_i - \operatorname{val}(c_0^i)} \binom{l}{j} \lambda_i^{l-j}. \tag{1}$$

The image of this in $\Bbbk$ is

$$\overline{c_j^{i+1}} = \frac{1}{j!} \frac{\partial^j f_i}{\partial x^j}(\lambda_i).$$

For $0 \leq j < r_{i+1}$ this is zero, since λ_i is a root of f_i of multiplicity r_{i+1}. For $j = r_{i+1}$ this is nonzero. Thus $\operatorname{val}(c_j^{i+1}) > 0$ for $0 \leq i \leq r_{i+1}$, and $\operatorname{val}(c_j^{i+1}) = 0$ for $j = r_{i+1}$. Note that we are using the fact that $\operatorname{char}(\Bbbk) = 0$ here.

If $c_0^{i+1} = 0$ then $x = 0$ is a root of F_{i+1}, so $\lambda_i t^{w_i}$ is root of F_i and so by recursing we get $\sum_{j=0}^{i} \lambda_i t^{w_0+\cdots+w_j}$ is a root of $F_0 = F$, and we are done. Thus we may assume that for each i we have $c_0^{i+1} \neq 0$, so F_{i+1} satisfies conditions 1 to 4 above, so we can continue.

The observation above on $\operatorname{val}(c_j^{i+1})$ implies that $k_{i+1} \leq r_{i+1} \leq k_i$. Since n is finite, the value of k_i can only drop a finite number of times, so there is $1 \leq k \leq n$ and m for which for $i \geq m$ we have $k_i = k$. This means that $r_i = k$ for all $i > m$, so $f_i = \mu_i(x-\lambda_i)^k$ for all $i > m$, and some $\mu_i \in \Bbbk$.

Let N_i be such that $c_j^i \in \Bbbk((t^{1/N_i}))$ for $0 \leq j \leq n$. We can take N_{i+1} to be the least common denominator of N_i and w_i by Equation 1. Let $y_i = \sum_{j=0}^{i} \lambda_i t^{w_0+\cdots+w_j} \in \Bbbk((t^{1/N_i}))$. We now show that we can take $N_{i+1} = N_i$ for $i > m$. In that case, we have $w_{i+1} = \operatorname{val}(c_0^i)/k$, so it suffices to show that for $i > m$ we have $\operatorname{val}(c_0^i) \in k/N_i\mathbb{Z}$. This follows from the fact that f_i is a pure power, so $\operatorname{val}(c_j^i) = (k-j)/k \operatorname{val}(c_0^j)$ for $1 \leq j \leq k$, and in particular $\operatorname{val}(c_{k-1}^i) = 1/k \operatorname{val}(c_0^j) \in 1/N_i\mathbb{Z}$. Thus there is an N for which $y_i \in \Bbbk((t^{1/N}))$ for all i, and so the limit

$$y = \sum_{j \geq 0} \lambda_j t^{w_0+\cdots+w_j} \in \Bbbk((t^{1/N})).$$

It remains to show that that y is a root of F. To see this, consider $z_i = \sum_{j \geq i} \lambda_j t^{w_i+\cdots+w_j}$, and note that $y = y_{i-1} + t^{w_0+\cdots+w_{i-1}} z_i$ for $i > 0$, so

$$F_i(z_i) = t^{\operatorname{val}(c_0^i)} F_{i+1}(z_{i+1}).$$

Since $z_0 = y$, it follows that

$$\operatorname{val}(F(y)) = \sum_{j=0}^{i} \operatorname{val}(c_0^j) + \operatorname{val}(F_{i+1}(z_{i+1})) > \sum_{j=0}^{i} \operatorname{val}(c_0^j)$$

for all $i > 0$. Since $\operatorname{val}(c_0^j) \in 1/N\mathbb{Z}$, we conclude that $\operatorname{val}(F(y)) = \infty$, so $F(y) = 0$ as required. □

If $\Bbbk$ has characteristic $p > 0$ then $\Bbbk\{\{t\}\}$ is not algebraically closed. This is because the Artin-Schreier polynomial $x^p - x - t^{-1}$ has p "roots" of the form

$$(t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \dots) + c$$

for c in the prime field $\mathbb{F}_p$ of $\Bbbk$. These are not Puiseux series, since there is no common denominator of the exponents, but do live in the field of *generalized power series* which we now define.

Fix an algebraically closed field $\Bbbk$, and a divisible group $G \subset \mathbb{R}$. The *Mal'cev-Neumann ring* $K = \Bbbk((G))$ of generalized power series is the set of formal sums $\alpha = \sum_{g \in G} \alpha_g t^g$ in an indeterminant t with the property that $\mathrm{supp}(\alpha) := \{g \in G : \alpha_g \neq 0\}$ is a well-ordered set.

If $\beta = \sum_{g \in G} \beta_g t^g$ then we set $\alpha + \beta = \sum_{g \in G} (\alpha_g + \beta_g) t^g$, and $\alpha\beta = \sum_{h \in G} (\sum_{g+g'=h} \alpha_g \beta_{g'}) t^h$. Then $\mathrm{supp}(\alpha + \beta) \subseteq \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$, so is well-ordered, and thus $\alpha + \beta$ is well-defined. For $\alpha\beta$, define $\mathrm{supp}(\alpha) + \mathrm{supp}(\beta)$ to be the set $\{g + g' : g \in \mathrm{supp}(\alpha), g' \in \mathrm{supp}(\beta)\}$. Then $\mathrm{supp}(\alpha)\,\mathrm{supp}(\beta)$ is well-ordered, and the set $\{(g, g') : g + g' = h\}$ is finite for all $h \in G$, so multiplication is well-defined.

The field of generalized power series is the most general field with valuation we need to consider in the following sense.

Theorem 3. *Fix a divisible group G and a residue field $\Bbbk$. Let K be a field with a valuation* val *with value group G such that* val *is trivial on the prime field ($\mathbb{F}_P$ or $\mathbb{Q}$) of K, and K has residue field $\Bbbk$. Then K is isomorphic to a subfield of $\Bbbk((t^G))$.*

One reference for these topics is [Poo93].

References

[Mar07] Thomas Markwig, *A field of generalized Puiseux series for tropical geometry*, 2007. arXiv:0709.3784.

[Poo93] Bjorn Poonen, *Maximally complete fields*, Enseign. Math. (2) **39** (1993), no. 1-2, 87–106. MR **1225257 (94h:**12005)