AARMS TROPICAL GEOMETRY - LECTURES 2 AND 3

DIANE MACLAGAN

The goal of today is to introduce affine and projective varieties. We adopt the simplistic motto that "Algebraic geometry is the study of solutions of polynomial equations". We are interested in the *geometry* of these solution spaces.

For example, consider the three equations $x^2+y^2-1=0$, $xy=0$, and $x^2+y^2=-1$. The first of these describes a circle of radius one, while the second is the union of two lines. The third has no solutions over the real numbers, but has solutions if work over the complex numbers (or at least an algebraically closed field), as we always will.

Definition 1. Let $\Bbbk$ be an algebraically closed field (such as $\mathbb{C}$). Then affine space is

$$\mathbb{A}^n = \{(a_1, \dots, a_n) : a_i \in \Bbbk\} = \Bbbk^n.$$

Philosophically $\mathbb{A}^n$ should be thought of as $\Bbbk^n$ without a distinguished origin.

Definition 2. Let $S = \Bbbk[x_1, \dots, x_n]$ be the polynomial ring in n variables with coefficients in $\Bbbk$. Given $f_1, \dots, f_s \in S$ the (affine) variety defined by the f_i is

$$V(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in \mathbb{A}^n : f_i(a_1, \dots, a_n) = 0 \text{ for } 1 \leq i \leq s\}.$$

Example: $V(x+y-1)$ is the line $y = x-1$.
Example: $V(x^2-y, x^3-z, y^3-z^2) = \{(t, t^2, t^3) : t \in \mathbb{C})\}$. This is the affine "twisted cubic" curve.
Note: $V(f_1, f_2) = V(f_1+f_2, f_1-f_2) = V(f_1, f_2, f_1+f_2, xf_1+y^2f_2)$.

Recall that an ideal $I \subseteq S$ is a set closed under addition and multiplication by elements of S. The ideal I generated by $f_1, \dots, f_s \in S$ is

$$I = \langle f_1, \dots, f_s \rangle = \{\sum_{i=1}^{s} g_i f_i : g_i \in S\}.$$

Lemma 3. *The variety $V(f_1, \dots, f_s)$ only depends on the ideal $\langle f_1, \dots, f_s \rangle$, so if $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_r \rangle$ then $V(f_1, \dots, f_s) = V(g_1, \dots, g_r)$.*

Thus we will talk about varieties as defined by ideals. Note that if the ideal is principal (generated by one element) then we call the variety a hypersurface. All the examples we saw yesterday were hypersurfaces.

Operations on varieties (The Ideal/Variety dictionary).

(1) $V(I) \cap V(J) = V(I+J)$. Here $I+J = \{f+g : f \in I, g \in J\}$. If $I = \langle f_1, \dots, f_s \rangle$, $J = \langle g_1, \dots, g_r \rangle$, then $I+J = \langle f_1, \dots, f_s, g_1, \dots, g_r \rangle$.
(2) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$, where $IJ = \{fg : f \in I, g \in J\}$. If $I = \langle f_1, \dots, f_s \rangle, J = \langle g_1, \dots, g_r \rangle$ then $IJ = \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq r \rangle$. A description of $I \cap J$ in terms of the f_i and g_j is not as simple (though there are algorithms to compute it).

Warning: We see here that we can have $V(I) = V(J)$ if $I \neq J$ (for example $IJ \neq I \cap J$ in general). For example,

$$V((x-y)^2) = V(x-y) = \{(a,a) : a \in \Bbbk\}.$$

Solution: Hilbert's Nullstellensatz.

Theorem 4. *Let $\Bbbk$ be an algebraically closed field. Then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$, where*

$$\sqrt{I} = \{f \in S : f^r \in I \text{ for some } r\}.$$

For a proof, see any book on commutative algebra (for example, [Eis95], or [CLO07]).
Example: If $I = \langle x^2 \rangle$, then $\sqrt{I} = \langle x \rangle$.

A subvariety of a variety $V(I)$ is a variety $V(J)$ with $V(J) \subset V(I)$. Note that if $V(J)$ is a subvariety of $V(I)$ then $\sqrt{I} \subseteq \sqrt{J}$.

We place a topology on $\mathbb{A}^n$ by setting the closed sets to be $\{V(I) : I \text{ is an ideal of } S\}$. This is the *Zariski topology.* To check that $\emptyset$ and $\mathbb{A}^n$ are closed, note that $\emptyset = V(1)$, and $\mathbb{A}^n = V(0)$. Exercise: Check that the finite union of closed sets and the arbitrary intersection of closed sets are closed. We denote by $\overline{U}$ the closure in the Zariski topology of a set U. This is the smallest set of the form $V(I)$ for some I that contains U.

Another operation on varieties

(3) $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$, where

$$I : J^\infty) = \{f \in S : \text{ for all } g \in J \text{ there exists } N > 0 \text{ with } fg^N \in I\}$$

is the *saturation* of the ideal I by the ideal J.

Important (for us) example: Let $J = \langle \prod_{i=1}^n x_i \rangle$. Then $V(J) = V(\prod_{i=1}^n x_i) = \cup_{i=1}^n V(x_i)$. For example, when $n = 2$, so $S = \mathbb{C}[x_1, x_2]$, then $J = \langle x_1 x_2 \rangle$, and $V(J)$ is the union of the two coordinate axes. The complement $\mathbb{A}^n \setminus V(J) = T^n = (\mathbb{C}^*)^n$, and for $I \in S$, $\overline{V(I) \setminus V(J)} = \overline{V(I) \cap T^n}$.
Example: $I = \langle x_1^2 + 3x_1x_2 \rangle$, $J = \langle x_1 x_2 \rangle$. Then $(I : J^\infty) = \{f \in S : \exists N \text{ such that } f x_1^N x_2^N \in I\rangle = \langle x_1 + 3x_2 \rangle$.

Definition 5. A variety X is *irreducible* if it cannot be written as the union of two proper subvarieties. This is a topological notion.

Proposition 6. *Let $X \subset \mathbb{A}^n$ be a variety. Then X can be written uniquely (up to order) as an irredundant union of irreducible varieties. These are called the irreducible components of X.*

Proof. We first show that such a decomposition exists. If X is irreducible then we are done. Otherwise we can write $X = X_1 \cup X_2$ where the X_i are proper subvarieties. Given a variety Y we write $I(Y)$ for the radical ideal defining Y. We must have $I(X) \subsetneq I(X_i)$ for $i = 1, 2$. Suppose now that we have a decomposition $I = \cup_{i=1}^s X_i^s$. If all of the X_i are irreducible, we are done. Otherwise there is some X_j that can be written in the form $X_j = X_j' \cup X_j''$ where X_j', X_j'' are proper subvarieties of X_j, so we replace X_j by X_j' and X_j'' in the decomposition and renumber to have $X_1^{s+1}, \dots, X_{s+1}^{s+1}$. In this fashion we can get a decreasing sequence $X_{i_1}^1 \supsetneq X_{i_2}^2 \supsetneq X_{i_3}^3 \supsetneq \dots$ s with corresponding increasing sequence $I(X_{i_1}^1) \subsetneq I(X_{i_2}^2) \subsetneq I(X_{i_3}^3) \subsetneq \dots$. Since S is

Noetherian this sequence must terminate at some stage s, at which point each X_i^s is irreducible, and $X = X_1^s \cup \cdots \cup X_s^s$ is an irreducible decomposition.

Now suppose that $X = X_1 \cup \cdots \cup X_s = Y_1 \cup \dots Y_r$ are two irredundant irreducible decompositions of X. Since $Y_1 \subset X$, $\cup_{i=1}^s (Y_1 \cap X_i) = Y_1$, so since Y_1 is irreducible there must be $1 \leq j_i \leq s$ with $Y_1 \subset X_{j_i}$. Similarly for all other Y_k there is j_k with $Y_k \subset X_{j_k}$. Reversing the roles of X_i and Y_i, we also get for each $1 \leq i \leq s$ there is l_i with $X_i \subseteq Y_{l_i}$. But this means that $Y_i \subseteq X_{j_i} \subseteq Y_k$ for $k = l_{j_i}$, so we must have $k = i$, so for each $1 \leq i \leq r$ there is k with $X_k = Y_i$. Since the decomposition in the X_i is irredundant, the X_i must equal to the Y_j up to order. □

Example: Let $I = \langle x_1^2 + 3x_1x_2 \rangle$. Then $I = \langle x_1 \rangle \cap \langle x_1 + 3x_2 \rangle$, so $V(I) = V(x_1) \cup V(x_1 + 3x_2)$. The varieties $V(x_1), V(x_1 + 3x_2)$ are both irreducible, so these are the irreducible components. The saturation of I by $J = \langle x_1x_2 \rangle$ removes the component $V(x_1)$ that does not intersect the torus.

Definition 7. The *coordinate ring* of X is $S/I(X)$. This is the ring of polynomial functions on X.

Projective Varieties.

Definition 8. Projective space $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \mathbf{0})/\sim$ where $\mathbf{v} \sim \lambda \mathbf{v}$ for all $\lambda \neq 0$. The points of $\mathbb{P}^n$ are the equivalence classes of lines through the origin $\mathbf{0}$. We write $[x_0 : x_1 : \cdots : x_n]$ for the equivalence class of $\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$. A line in $\mathbb{P}^n$ is the equivalence class of a two-dimensional subspace in $\mathbb{C}^{n+1}$.

We can think of $\mathbb{P}^n$ as $\mathbb{A}^n$ with some points "at infinity" added. For example, there is a bijection between $\mathbb{A}^n$ and the points of $\mathbb{P}^n$ of the form $[1 : x_1 : \cdots : x_n]$. The "points at infinity" are then those with first coordinate 0.

Example: $\mathbb{P}^2 = (\mathbb{C}^3 \setminus (0,0,0))/\sim$. Then $\mathbb{P}^3 = \{[1 : x_1 : x_2] : (x_1, x_2) \in \mathbb{A}^2\} \cup \{[0 : x_1 : x_2] : x_1, x_2 \in \mathbb{C}^2\} = \{[1 : x_1 : x_2] : (x_1, x_2) \in \mathbb{A}^2\} \cup \{[0 : 1 : x_2] : x_2 \in \mathbb{C}\} \cup \{[0 : 0 : 1]\}$. So we see that the points at infinity are a copy of $\mathbb{P}^1$.

Note: Polynomials don't make sense as functions on $\mathbb{P}^n$. For example, $[1 : 2 : 3] = [2 : 4 : 6] \in \mathbb{P}^2$, but the function $x_1 + x_2$ has different values (5 or 10) on these two points. However if $f \in S$ is homogeneous, then $\{x \in \mathbb{P}^n : f(x) = 0\}$ is well-defined. This is because if $f(x) = 0$, then $f(\lambda x) = 0$ for all $\lambda \neq 0$, since if f is homogeneous of degree k, then $f(\lambda x) = \lambda^k f(x)$.

We call an ideal $I \subset \Bbbk[x_0, \dots, x_n]$ *homogeneous* if it has a homogeneous generating set.

Definition 9. Let I be a homogeneous ideal in $S = \Bbbk[x_0, \dots, x_n]$. Then the variety of I is

$$V(I) = \{[x] \in \mathbb{P}^n : f(x) = 0 \ \ \forall [x] \in \mathbb{P}^n\}.$$

Example: $V(x_0 + x_1 + x_2) = \{[1 : t : -1 - t] : t \in \mathbb{C}\} \cup \{[0 : 1 : -1]\}$.

Example: $V(x_0, x_1, x_2) = \emptyset$.

The same rules apply for varieties in $\mathbb{P}^n$ as for $\mathbb{A}^n$:

(1) $V(I) \cap V(J) = V(I + J)$;

(2) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$;

(3) $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$.

As in the affine case, there is not a bijection between homogeneous ideals and projective varieties. Let $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$. We call $\mathfrak{m}$ the "irrelevant ideal", as it is the largest ideal not corresponding to a nonempty subvariety of $\mathbb{P}^n$.

Lemma 10. *Let $V(I), V(J)$ be subvarieties of $\mathbb{P}^n$. Then $V(I) = V(J) \neq \emptyset$ if and only if*

$$\sqrt{I} = \sqrt{J}$$

. Also, $V(I) = \emptyset$ if and only if $I = \langle 1 \rangle$ or $\sqrt{I} = \mathfrak{m}$.

Proof. We first consider the case $V(I) \neq \emptyset$. Let $\widetilde{V(I)}, \widetilde{V(J)}$ denote the subvarieties of $\mathbb{A}^{n+1}$ defined by I and J. Note that if $x \in \widetilde{V(I)}$ then $\lambda x \in \widetilde{V(J)}$ for all $\lambda \neq 0$ (and similarly for $V(J)$). Thus $V(I) = V(J)$ if and only if $(\widetilde{V(J))} \setminus \{\mathbf{0}\}) = (\underline{\widetilde{V(J))} \setminus \{\mathbf{0}}\})$. Now if $f \in S$ satisfies $f(\lambda x) = 0$ for all $x \in \widetilde{V(I)} \setminus \mathbf{0}$ then $f(\mathbf{0}) = 0$, so $\widetilde{V(I)} \setminus \mathbf{0} = \widetilde{V(I)}$. Thus $\widetilde{V(I)} \setminus \mathbf{0} = \widetilde{V(J)} \setminus \mathbf{0}$ if and only if $\sqrt{I} = \sqrt{J}$ by the Nullstellensatz.

Also $V(I) = \emptyset$ if and only if $\widetilde{V(I)} = \emptyset$ or $\widetilde{V(I)} = \{\mathbf{0}\}$, so if and only if $I = \langle 1 \rangle$ or $\sqrt{I} = \mathfrak{m}$. $\square$

Definition 11. The homogeneous coordinate ring of a projective variety $X = V(I)$ is S/I, where $S = \Bbbk[x_0, \ldots, x_n]$.

Subvarieties of tori.

The last case of varieties that we will consider is that of subvarieties of tori. This is actually a special case of affine varieties. Let $S = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of Laurent polynomials.

Example: $f = 3x_1x_2^2 + 5x_1x_2^3 + 7x_1^{-5}x_2 \in S$.

The ring S is the coordinate ring of the *algebraic torus* $T^n \cong (\Bbbk^*)^n$. The name comes from the fact that $(\mathbb{C}^*)^n$ deformation retracts to the standard topological n-dimensional torus $(S^1)^n$. The ring S is the ring of all those rational functions (quotients of polynomials) that are defined everywhere on T^n.

An ideal $I \subset S$ determines a subvariety

$$V(I) = \{\mathbf{x} \in T^n : f(x) = 0 \text{ for all } f \in I\} \subseteq T^n.$$

Note that it makes sense to consider $f(x)$ for $x \in T^n$, since any $f \in S$ has the form $g/(\prod_{i=1}^n x_i)^N$ for some polynomial g and $N \geq 0$, so is defined at any $x \in T^n$.

Note that a subvariety of T^n is actually also an affine variety, which can be embedded into $\mathbb{A}^{n+1}$. If $X = V(I) \subset T^n$, choose a generating set for I consisting of polynomials $\{f_1, \ldots, f_s\}$. This can always be done, since every monomial is a unit in S. Let S' be the polynomial ring $\Bbbk[x_1, \ldots, x_n, y]$, and let J be the ideal $\langle f_1, \ldots, f_s, y\prod_{i=1}^n x_i - 1\rangle$, where we consider the f_i here as elements of S'. Then the affine variety of J in $\mathbb{A}^{n+1}$ consists of the points $\{(x, 1/\prod_{i=1}^n x_i) \in \mathbb{A}^{n+1} : x \in V(I) \subset T^n\}$.

Warning: You'll notice I'm using S for three different rings here: $S = \Bbbk[x_1, \ldots, x_n]$, the coordinate ring of $\mathbb{A}^n$; $S = \Bbbk[x_0, \ldots, x_n]$, the homogeneous coordinate ring of $\mathbb{P}^n$; and $S = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the coordinate ring of T^n. The meaning should always be clear from context, and this has the advantage that one can summarize the previous discussion in the following form:

Note also that in each case there is a largest ideal defining a variety X, which we denote by $I(X)$. For example if $X = V(I) \subseteq \mathbb{A}^n$, then $I(X) = \sqrt{I}$.

Summary:

Let X, Y be subvarieties of $\mathbb{A}^n$, $\mathbb{P}^n$ or $\mathbb{T}^n$, with ideals $I(X), I(Y)$ in the respective coordinate ring. Then

(1) $I(X \cup Y) = I(X) \cap I(Y) = I(X)I(Y)$;
(2) $I(X \cap Y) = I(X) + I(Y)$;
(3) $I(\overline{X \setminus Y}) = I(X) : I(Y)^\infty$;
(4) The coordinate ring of X is $S/I(X)$.

Dimension

We will study many invariants of a variety X. A basic one is the *dimension* of X. We first give an intuitive definition of dimension. Nice ("smooth" or "nonsingular") complex varieties are real manifolds of dimension $2d$ for some integer d. We say that the (complex) dimension of such an X is d.

Example: The projective variety $\mathbb{P}^1$ is equal to the two-dimensional sphere S^2 as a set. This has real dimension two as a manifold, so the dimension of $\mathbb{P}^1$ is one.

Saying that $\dim(X) = d$ is intuitively saying that near most points X looks like $\mathbb{C}^d$. (Intentionally vague sentence!)

Formally, the dimension of an irreducible variety X is the length d of the longest chain

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d = X$$

of irreducible subvarieties. (Note that this definition works for subvarieties of $\mathbb{A}^n, \mathbb{P}^n, T^n$.)

Example: $\{V(x_1, x_2) = (0,0)\} \subsetneq V(x_1) \subsetneq \mathbb{A}^2$, so $\dim(\mathbb{A}^2) \geq 2$. In fact $\dim(\mathbb{A}^2) = 2$ (and $\dim(\mathbb{A}^n) = n$ for all n, but this is (surprisingly?) not trivial.

Example: $\{(1,1) = V(x_1 - 1, x_2 - 1) \subsetneq V(x_1^2 + x_2^2 - 1)$, so the dimension of $V(x_1^2 + x_2^2 - 1)$ is at least one. Again, in this case it is exactly one.

There is an equivalent algebraic definition of dimension. The *Krull dimension* of a ring R is the length d of the longest chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d = R$$

of prime ideals. If $X \subset \mathbb{A}^n$ or $X \subset T^n$ then $\dim(X)$ is the Krull dimension of the coordinate ring $S/I(X)$. If $X \subset \mathbb{P}^n$ then $\dim(X)$ is one less than the dimension of $S/I(X)$. For an overview of Krull dimension, see [Eis95, Chapter 8].

Intersection multiplicity.

An affine variety (or subvariety of a torus) is zero-dimensional if and only if $\dim_{\mathbb{k}} S/I$ is finite. When two affine varieties X and Y have a zero-dimensional intersection, we will be interested in the *multiplicity* of this intersection. This is the number $\dim_{\mathbb{k}} S/(I(X) \cap I(Y))$.

Example: $I = \langle x_1^2 + x_2 \rangle$, $J = \langle x_2 \rangle \subset S = \mathbb{k}[x_1, x_2]$. Then $I \cap J = \langle x_1^2, x_2 \rangle$, so the multiplicity is $\dim_{\mathbb{k}} \mathbb{k}[x_1, x_2]/\langle x_1^2, x_2 \rangle = 2$.

Notes

(1) You may recognize this number from Bézout's theorem, one form of which says that when the variety in $\mathbb{P}^n$ defined by n polynomials $f_1, \ldots, f_n$ of degrees

$d_1, \ldots, d_n$ is zero-dimensional, then it consists of $\prod_{i=1}^n d_i$ points, counted with multiplicity.

(2) In fancier language, the multiplicity is the length of the scheme-theoretic intersection of I and J. We will not use the language of schemes this month, but as a (very!) first approximation take the word "scheme-theoretic" to mean "take the intersection of the defining ideals, without taking the radical".

(3) When $V(I), V(J)$ are projective varieties, we replace the multiplicity by the (scheme-theoretic) *degree* of the zero-dimensional intersection. This is the number $\dim_{\Bbbk}(S/I \cap J)_l$ for $l \gg 0$, where $(S/I \cap J)_l$ is the degree lth graded piece of the graded S-module $S/I \cap J$.

References

[CLO07] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, 3rd ed., Undergraduate Texts in Mathematics, Springer, New York, 2007. An introduction to computational algebraic geometry and commutative algebra.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.