AARMS TROPICAL GEOMETRY - LECTURE 16

DIANE MACLAGAN

Summary of Class:

We begin this lecture by summarizing what we have done in this class.

We have constructed a *tropical variety* $\operatorname{trop}(X)$ associated to a variety $X \subset T^n$ with $X = V(I)$ for $I \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. This was defined by

$$\operatorname{trop}(X) = \bigcap_{f \in I} \operatorname{trop}(V(f)),$$

where for $f = \sum_{v \in \mathbb{Z}^n} c_v x^v$ the set $\operatorname{trop}(V(f))$ is the set of $w \in \mathbb{R}^n$ for which the tropicalization $\operatorname{trop}(f)(w) = \min(\operatorname{val}(c_v) + w \cdot v)$ is achieved at least twice. We have the Fundamental Theorem of Tropical Algebraic Geometry:

Theorem 1. *Let $X = V(I) \subset T^n$ for $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the following sets coincide:*

(1) $\operatorname{trop}(X)$*;*
(2) *The closure in $\mathbb{R}^n$ of $\{w \in (\operatorname{im} \operatorname{val})^n : \operatorname{in}_w(I) \neq \langle 1 \rangle\}$.*
(3) *The closure in $\mathbb{R}^n$ of $\{\operatorname{val}(y) : y \in X\}$.*

The last part of the Fundamental Theorem means we can think of a tropical variety as a *combinatorial shadow* of the variety $X \subset T^n$. We saw that tropical varieties have the following structure:

Theorem 2 (Structure Theorem for Tropical Varieties)**.** *Let $X \subset T^n$ be an irreducible variety of dimension d. Then* $\operatorname{trop}(X)$ *is the support of a balanced weighted polyhedral complex that is pure of dimension d that is connected in codimension one.*

We saw this in examples we computed with `gfan`, and in the specific explicit examples of linear varieties and $G(2, n)$.

In the spirit of the "combinatorial shadow" philosophy, the (somewhat philosophical) question guiding work in this form of tropical geometry is:

Question: Which invariants of X can be computed from $\operatorname{trop}(X)$?

For example, we see from the main structure theorem that $\dim(X) = \dim(\operatorname{trop}(X))$, so if we know the tropical variety we know the dimension. An answer to the guiding philosophical question is a current area of active research.

Recall that a variety $\mathbb{P}$ is a *toric variety* if $\mathbb{P}$ contains a dense copy of T^n, and there is an action of T^n on $\mathbb{P}$ extending the action of $\mathbb{T}^n$ on itself. A polyhedral fan Σ defines a toric variety $\mathbb{P}_\sigma$. The cones of Σ index the T^n-orbits of $\mathbb{P}_\sigma$. Examples include T^n, $\mathbb{A}^n$, $\mathbb{P}^n$, and $\mathbb{P}^1 \times \mathbb{P}^1$.

Often we are interested not in some $X \subset T^n$¡ but in a projective variety $Y \subset$ $mathbbP^m$ for some m. Tropical geometry can be useful if this occurs in the following way.

(1) Choose $X \subseteq Y$ and an embedding $X \subset T^n$ for some n.

(2) Let $\operatorname{trop}(X)$ be the fan $\Sigma \subset \mathbb{R}^n$, and let $\mathbb{P}_\Sigma$ be the toric variety with fan Σ.
(3) Let $\overline{X}$ be the closure of X in $\mathbb{P}_\Sigma$.
(4) If $Y \cong \overline{X}$ then we can hope to learn about $Y = \overline{X}$ from $\operatorname{trop}(X)$.

A variety Y that arises in this fashion is called a *tropical compactification* of X. This was first considered in the work of Tevelev [Tev07] with extra conditions on the fan structure on Σ, which guarantees nicer properties for the geometry of $\overline{X}$.

One of the main invariants of $Y = \overline{X}$ we can hope to learn from $\operatorname{trop}(X)$ is the Chow ring of Y, which is the algebraic version of the cohomology ring. This area of algebraic geometry is known as intersection theory. Those unfamiliar with the cohomology ring should spend some time with an algebraic topology book, such as [Hat02] (freely electronically available from the author's webpage). The enumeration of curves problem we will discuss next is such a form of intersection problem.

Other aspects of tropical geometry

This philosophy of obtaining information about $X \subset T^n$, or a projective compactification $\overline{X}$, from the tropical variety $\operatorname{trop}(X)$ is only one portion of current research in tropical geometry. Some samples of this philosophy are found in the work of Speyer [DS05], Payne [Pay07], Tevelev [Tev07], and Hacking, Keel, and Tevelev [HKT06]. The key ideas behind the Fundamental Theorem arose from the work of Kapranov, which appears in [EKL06]. We emphasize that these references are just a (nonrepresentative) sample of the literature.

We now give some ideas of other aspects of tropical geometry. This is only a brief overview, and references will just be a nonrepresentative sample.

Abstract tropical varieties

We can *define* a tropical variety to be a balanced weighted polyhedral complex Σ, and then define tropical versions of algebraic geometry invariants by analogy with their classical definitions. An example of this includes defining the dimension to be the dimension of the complex Σ. Current work (see [LAJR07]) is developing a version of intersection theory in this context. Another possibility would be a tropical version of irreducible components. This creates difficulties, though, as the following example shows.

Example: Let $X_1 = V((x+y+1)(x+y+xy)) = V(x+y+1) \cup V(x+y+xy) \subset T^2$. Then $\operatorname{trop}(X_1) = \operatorname{trop}(x+y+1) \cup \operatorname{trop}(x+y+xy)$ is shown in Figure 1. Let $X_2 = V((x-1)(y-1)(x-y)) = V(x-1) \cup V(y-1) \cup V(x-y)$. Then $\operatorname{trop}(X_2)$ is the union of the lines $w_1 = 0$, $w_2 = 0$ and $w_1 = w_2$, so equals $\operatorname{trop}(X_2)$. Since we can't write any of these last three lines as the union of smaller balanced weighted polyhedral complexes, this means that $\operatorname{trop}(X_1) = \operatorname{trop}(X_2)$ cannot be written uniquely as the union of "irreducible" tropical varieties, so a naive definition of irreducible components of these "abstract" tropical varieties does not work.

Similarly, we will see later in the week that there are three different notions of the ranks of a matrix over the tropical semiring. One of the motivations for understanding which geometric invariants make sense for these abstract tropical varieties is that it indicates which properties we can expect combinatorial proofs and interpretations of. This can be thought of as the synthetic approach to tropical geometry. Some

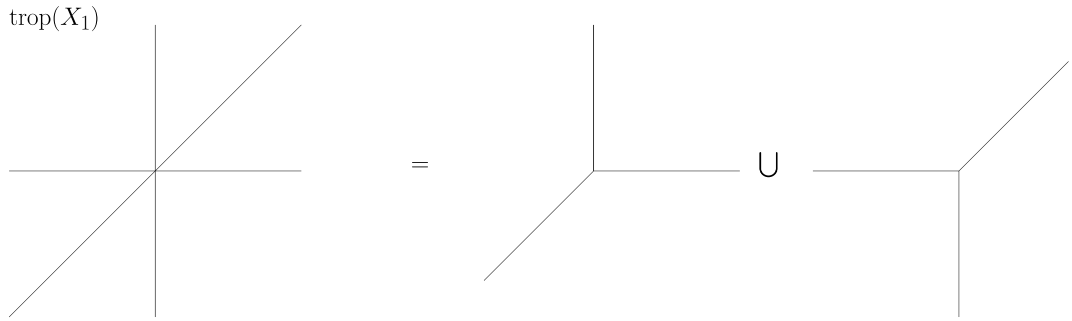

FIGURE 1.

examples can be seen in the work of Gathmann and his students, such as [GK08], [LAJR07].
Warning: Note every balanced weighted polyhedral complex in $\mathbb{R}^n$ is trop(X) for some $X \subset T_K^n$. See the example of Mikhalkin in [DS05, Figure 5.1].
Open Question: Characterize those balanced weighted polyhedral complexes in $\mathbb{R}^n$ that are trop(X) for some $X \subset T_K^n$.

More general tropical geometry

Another vital branch of tropical research can be broadly classified as attempting to do *all* geometry, not merely algebraic geometry, over the tropical semifield. For example, there should be tropical manifolds, and complex and differential geometry in this setting. This the approach currently taken by Mikhalkin to establish the basics of tropical geometry. See [Mik06], [GM07], [GM] for some expositions of this approach. This also leads to intersections and applications for real algebraic geometry; see, for example, [IKS05]. See [IMS07] for more in this line.

Other aspects

Another major school of tropical geometry revolves around the approach of Gross and Siebert to mirror symmetry using tropical geometry; see [GS06] and its references.

Part of the excitement of this field is that new areas and applications are constantly arising. For example, Gubler [Gub07] recently used the connections to rigid analytic geometry and Berkovich spaces for applications in number theory. Finally, as well as the emerging tropical geometry community, there is also research in max-plus algebras in control theory and related topics, which is a more established field.

These references are only a sample of the exciting research happening in tropical geometry. For more, google "tropical geometry", or put "tropical" in the "anywhere" search box at `front.math.ucdavis.edu` or `mathscinet`.

REFERENCES

[LAJR07] Lars Allermann and Johannes Rau, *First Steps in Tropical Intersection Theory*, 2007. arXiv:0709.3705.

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157. MR **2289207 (2007k:**14038)

[GK08] Andreas Gathmann and Michael Kerber, *A Riemann-Roch theorem in tropical geometry*, Math. Z. **259** (2008), no. 1, 217–230. MR 2377750

[GS06] Mark Gross and Bernd Siebert, *Mirror symmetry via logarithmic degeneration data. I*, J. Differential Geom. **72** (2006), no. 2, 169–338. MR **2213573 (2007b:**14087)

[Gub07] Walter Gubler, *Tropical varieties for non-Archimedean analytic spaces*, Invent. Math. **169** (2007), no. 2, 321–376. MR 2318559

[HKT06] Paul Hacking, Sean Keel, and Jenia Tevelev, *Compactification of the moduli space of hyperplane arrangements*, J. Algebraic Geom. **15** (2006), no. 4, 657–680. MR **2237265 (2007j:**14016)

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR **1867354 (2002k:**55001)

[IKS05] Ilia Itenberg, Viatcheslav Kharlamov, and Eugenii Shustin, *Logarithmic asymptotics of the genus zero Gromov-Witten invariants of the blown up plane*, Geom. Topol. **9** (2005), 483–491 (electronic). MR **2140988 (2006i:**14059)

[IMS07] Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin, *Tropical algebraic geometry*, Oberwolfach Seminars, vol. 35, Birkhäuser Verlag, Basel, 2007. MR **2292729 (2008e:**14082)

[Mik06] Grigory Mikhalkin, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 827–852. MR **2275625 (2008c:**14077)

[GM07] Grigory Mikhalkin, *Introduction to Tropical Geometry (notes from the IMPA lectures in Summer 2007)*, 2007. arXiv:0709.1049.

[GM] Grigory Mikhallkin, *Tropical Geometry*. Book in progress. See `http://www.math.toronto.edu/mikha/TG-project.html`.

[Pay07] Sam Payne, *Fibers of tropicalization*, 2007. arXiv:0705.1732. To appear in Math. Z.

[DS05] David Speyer, *Tropical Geometry*, UC Berkeley, 2005.

[Tev07] Jenia Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (2007), no. 4, 1087–1104. MR **2343384 (2008f:**14068)