AARMS TROPICAL GEOMETRY - LECTURE 12

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In this lecture we discuss the tropicalization of the Grassmannian. We first review this beautiful classical variety.

Let V be an n-dimensional vector space. We will use $\mathbb{k}$ to denote the field of V by default, but the constructions will make sense over an arbitrary vector space. The Grassmannian $G(d,n)$ parameterizes all d-dimensional subspaces of $V \cong \mathbb{k}^n$. It is a smooth projective variety of dimension $d(n-d)$ (so "locally" looks like affine space of dimension $d(n-d)$), whose points correspond to d-dimensional subspaces of V.

Example: The Grassmannian $G(1,n)$ parameterizes all one-dimensional subspaces of V, which are lines through the origin in $\mathbb{k}^n$. Thus $G(1,n) \cong \mathbb{P}^{n-1}$.

We'll describe two ways to see that $G(d,n)$ is a variety.

Approach 1: We'll describe $G(d,n)$ by giving an *affine cover*. An affine cover is open cover (in the Zariski topology) of the set $G(d,n)$ by affine varieties. It is analogous to giving charts for a manifold.

Fix a basis for V, and thus an isomorphism of V with $\mathbb{k}^n$. Given a d-dimensional subspace $W \subseteq V \cong \mathbb{k}^n$, choose a basis for W and write this basis, in the basis for V, as the rows of a $d \times n$ matrix W_{mat} with entries in $\mathbb{k}$. For each set $\sigma \subset \{1, \dots, n\}$ of size d we consider the set

$$G_\sigma = \{W \subset V : \text{ the submatrix of } W_{mat} \text{ indexed by the columns of } \sigma \text{ is invertible}\}.$$

Given $W \in G_\sigma$, there is a unique basis for W for which $W_{mat}|_\sigma$ is the identity matrix.

Example: Consider $G(2,4)$, and $\sigma = \{1,2\}$. Then if $W \in G_\sigma$, then there is a choice of basis for W in which

$$W_{mat} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}, \tag{1}$$

where $a, b, c, d \in \mathbb{k}$. So $G_\sigma \cong \mathbb{A}^4$. We can glue together all the different G_σ to get a variety.

Example: $G_{12} \cap G_{13}$. Then if $W \in G_{12} \cap G_{13}$ then there is a choice of basis for W as above, where $c \neq 0$. Thus $G_{12} \cap G_{13} \cong \mathbb{A}^3 \times \mathbb{k}^*$. There is also a version of W_{mat} of the form

$$W_{mat} = \begin{pmatrix} 1 & -a/c & 0 & b - ad/c \\ 0 & 1/c & 1 & d/c \end{pmatrix}.$$

We identify the two copies of $\mathbb{A}^4$ on their overlap using the above transitions.

Approach 2: We can embed $G(d,n)$ into $\mathbb{P}^{\binom{n}{d}-1}$ by sending $W \in G(d,n)$ to the vector of $d \times d$ minors of W_{mat}. This is the Plücker embedding of $G(d,n)$. Note that if we choose a different basis for W, then the vector of minors changes by at most a multiplicative constant, so the map to $\mathbb{P}^{\binom{n}{d}-1}$ is well-defined. This map is injective, and the image is a subvariety cut out by the *Plucker relations*, which we explain in the case $d = 2$.

Example: Consider $G(2,4)$. Then $\binom{4}{2} = 6$. Label the coordinates of $\mathbb{P}^5$ as $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$. The Plücker relation is $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$. For example, for the matrix W_{mat} of Equation 1, the Plücker image is $(1 : c : d : -a : -b : ad - bc)$. This satisfies the Plücker relation.

Theorem 1. *Let*

$$I_{2,n} = \langle p_{ijkl} = x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} : 1 \leq i < j < k < l \leq n \rangle \subset \Bbbk[x_{ij} : 1 \leq i < j \leq n].$$

Then $G(2,n) = V(I_{2,n}) \subset \mathbb{P}^{\binom{n}{2}-1}$.

There is a similar description for $G(d,n)$ for $d > n$.

Let $AG(2,n) = V(I_{2,n}) \subset \mathbb{A}^{\binom{n}{2}}$. Then $AG(2,n)$ is the *affine cone* over the Grassmannian $G(2,n)$. Let T be torus of $\mathbb{A}^{\binom{n}{2}}$. Then $T = \{(x_{12}, x_{13}, \ldots, x_{(n-1)n}] \in \mathbb{A}^{\binom{n}{2}} : x_{ij} \neq 0\}$. Let $X = AG(2,n) \cap T$.

Claim: The set of Plücker relations forms a tropical basis for $I_{2,n} \subseteq K[x_{ij}^{\pm 1}]$, so

$$\operatorname{trop}(X) = \bigcap_{1 \leq i < j < k < l \leq n} \operatorname{trop}(V(p_{ijkl})).$$

For a proof, see [SS04].

Thus $\operatorname{trop}(X) = \{w \in \mathbb{R}^{\binom{n}{2}} :$ for all $1 \leq i < j < k < l \leq m$ either $w_{ij} + w_{kl} = w_{ik} + w_{jl} \leq w_{il} + w_{jk}$, or $w_{ij} + w_{kl} = w_{il} + w_{jk} \leq w_{ik} + w_{jl}$, or $w_{ik} + w_{jl} = w_{il} + w_{jk} \leq w_{ij} + w_{kl}\}$. This is the *space of Phylogenetic trees.*

Definition 2. A *tree* is a graph with no cycles. The *degree* of a vertex in a tree is the number of edges incident to that vertex. A tree is *trivalent* if every vertex has degree three or one. The vertices of degree one are called *leaves.* A tree is *leaf-labelled* if all leaves have a label from some set S. A *phylogenetic tree* is a trivalent leaf-labelled tree.

The name comes from the fact that diagrams illustrating closeness of species are trivalent graphs. See Figure 1.

Given a phylogenetic tree τ drawn in the plane, we can record the *tree-distance* between two vertices.

Example: For the tree of Figure 2, the tree-distance between vertices 1 and 2 is $a + b$. Ordering the pairs (i,j) as $12, 13, 14, 23, 24, 34$, we have the distance between vertices i and j recorded in the vector

$$(a + b, a + c + d, a + c + e, b + c + d, b + c + e, d + e).$$

This equals

$$a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + e \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

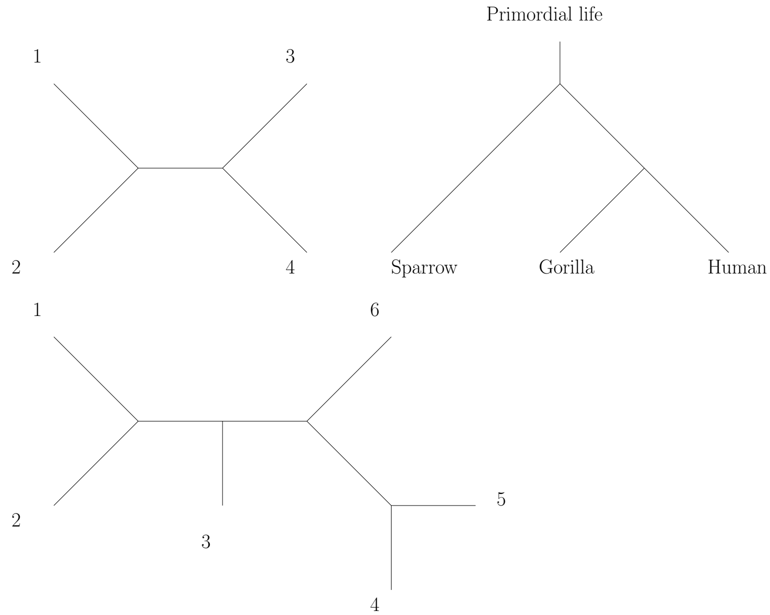

Figure 1. Some phylogenetic trees

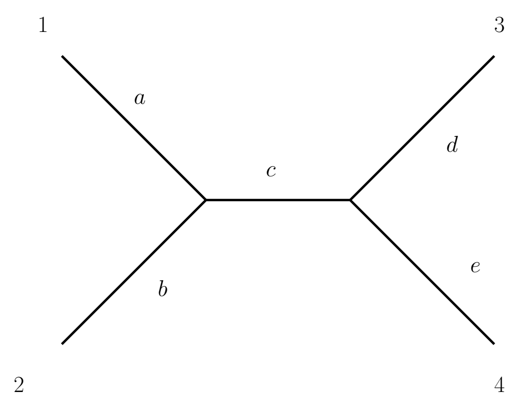

Figure 2.

If we let the leaf lengths become negative, this cone is then

$$\operatorname{span}\left(\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\1\\1\end{pmatrix}\right)+\operatorname{pos}\left(\begin{pmatrix}0\\1\\1\\1\\1\\0\end{pmatrix}\right),$$

which equals

$$\{w : w_{13}+w_{24}=w_{14}+w_{23}\geq w_{12}+w_{34}\}.$$

In general, a vector is the tree-distance vector for a trivalent tree with n leaves if and only if it satisfies the *four point condition:* for any four leaves i,j,k,l, if we form the three possible sums of distances between disjoint pairs $d(i,j)+d(k,l)$,

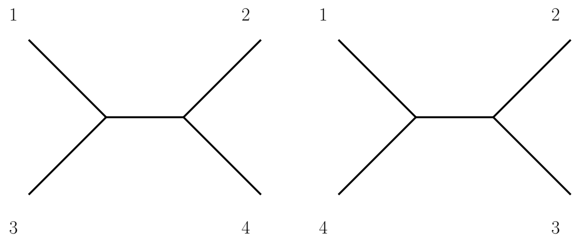

FIGURE 3.

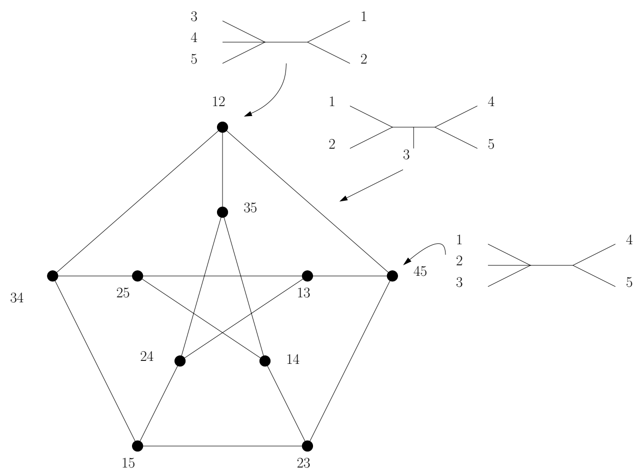

FIGURE 4.

$d(i,k)+d(j,l)$, and $d(i,l)+d(j,k)$, then two of these sums are equal and greater than or equal to the third.

For trees with four leaves the other two options are shown in Figure 3. Setting

$$V = \text{span}(\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\\1\end{pmatrix}),$$

these cones are $V + \text{pos}((1,0,1,1,0,1))$ and $V + \text{pos}((1,1,0,0,1,1))$.

Note that the union of these three cones is $-\operatorname{trop}(X)$!

Summary: The set $-\operatorname{trop}(X)$ is the set of tree-distance vectors for phylogenetic trees with n leaves. This is the *space of phylogenetic trees*, and has one cone for each combinatorial type of labelled tree. It is a polyhedral fan, with an n-dimensional lineality space. The quotient by the lineality space is a pure fan of dimension $n-3$. Note that $n-3$ is the number of internal (not adjacent to a leaf) edges of a trivalent tree with n leaves.

When $n=4$ we get the three cones described above. When $n=5$ the quotient by the five-dimensional lineality space is a two-dimensional fan in $\mathbb{R}^5$. The intersection of this with the sphere S^4 is the graph shown in Figure 4.

For more details on all this, see [SS04].

References

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411. MR **2071813 (2005d:**14089)