AARMS TROPICAL GEOMETRY - LECTURE 11

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The goal of this lecture is to describe how to compute tropical varieties in general.

We begin with an example of the algorithm outlined for linear varieties last lecture.

Example: Let $X = V(x_0 - x_1 + x_3, x_0 - x_2 + x_4, x_1 - x_2 + x_5) \subset T^6$. Then

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

If we add the additional first row of $(1,1,1,0,0,0)$ to B, which lies in the row space of B, then the columns of B are the positive roots of the root lattice of A_3. We have

$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

The lattice of flats $\mathcal{L}(\mathcal{B})$ is then illustrated in Figure 1. Here 013 means $\operatorname{span}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_3)$. The order complex then has 18 cones, which are indexed by the edges in Figure 1 connecting two proper nontrivial flats. The cones in $\operatorname{trop}(X) \subset \mathbb{R}^6$ are then three-dimensional, as expected. When we quotient by the lineality space $\operatorname{span}(\mathbf{1})$, we get a two-dimensional fan in $\mathbb{R}^5 \cong \mathbb{R}^6/\operatorname{span}(\mathbf{1})$. Intersecting this with the sphere $S^4 \subset \mathbb{R}^5$ gives a the graph that is shown in Figure 2. This is a refinement (subdividing the edges 05, 14, and 23) of a well-studied graph called the *Peterson graph*.

We now describe how to use a computer to compute tropical varieties. That first raises the following question:

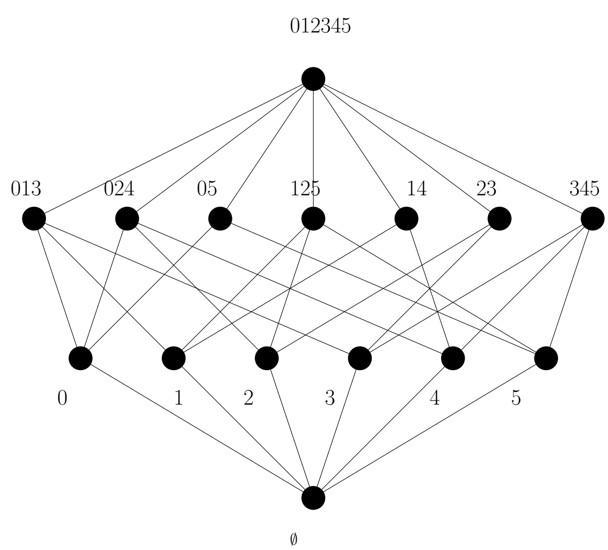

FIGURE 1.

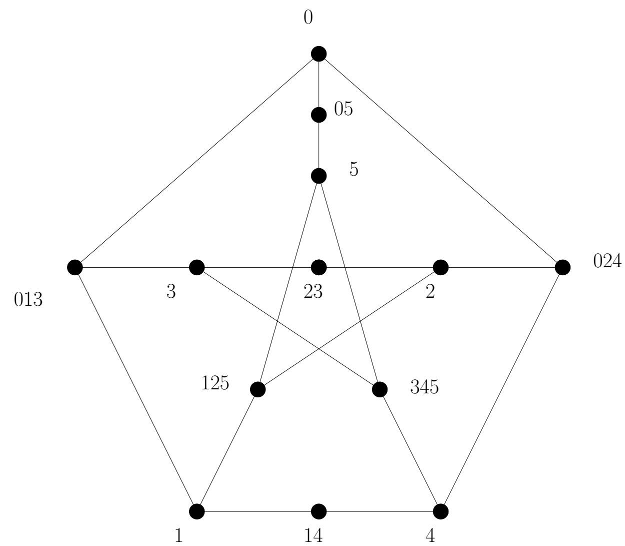

FIGURE 2.

Question: What sort of examples can we expect to do on a computer?

We cannot enter a general Puiseux series into a computer, as it cannot be described by a finite amount of information. Similarly, we cannot describe most real numbers in finite space. This suggests that the best field we can hope to work with is the algebraic closure $\overline{\mathbb{Q}(t)}$ of the ring of rational functions in t with coefficients in $\mathbb{Q}$. Note that every example we have seen so far as lived in this field!

Any ideal in $\overline{\mathbb{Q}(t)}$ actually lives in some finite extension L of $\mathbb{Q}(t)$, since the ideal is finitely generated, and a finite generating set has only a finite number of coefficients. We can write $L \cong \mathbb{Q}(t)[a_1, \dots, a_s]/J$, where J is an ideal. As a point of comparison, recall that $\mathbb{C} = \mathbb{R}[x]/\langle x^2+1\rangle$. We'll now discuss how to reduce this case to computing in $\mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

If $I \subset \mathbb{Q}(t)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, then let $J = I \cap \mathbb{Q}[t^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Fix $w \in \mathbb{R}^n$. Then $w \in \operatorname{trop}(V(I)) \subset \mathbb{R}^n$, where $V(I) \subset T^n$, if and only if $(1, w) \in \operatorname{trop}(V(J) \subset \mathbb{R}^{n+1}$, where $V(J) \subset T^{n+1}$. This is the case because if $p(t)$ is a polynomial in t, then $\operatorname{val}(p(t))$ is the smallest exponent of t occuring in p.

If $I \subset L[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ where $L = \mathbb{Q}(t)[a_1, \dots, a_s]/J$ is a finite extension of $\mathbb{Q}(t)$, then let $v_i = \operatorname{val}(a_i)$. Consider the map

$$\phi\colon \mathbb{Q}[a_1^{\pm 1}, \dots, a_s^{\pm 1}, t^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to L[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Fix $w \in \mathbb{R}^n$. Then $w \in \operatorname{trop}(V(I))$ if and only if $(v_1, \dots, v_s, 1, w) \in \operatorname{trop}(V(J))$, where $J = \phi^{-1}(I)$, and $V(J) \subset T^{n+s+1}$.

Example: Let $f = 1 + x + xy + ty \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ or in $\overline{Q(t)}[x^{\pm 1}, y^{\pm 1}]$. Then $\operatorname{trop}(V(f))$ is shown in Figure 3. Let $g = 1 + x + xy + ty \in \overline{Q}[t^{\pm 1}, x^{\pm 1}, y^{\pm 1}]$. Then

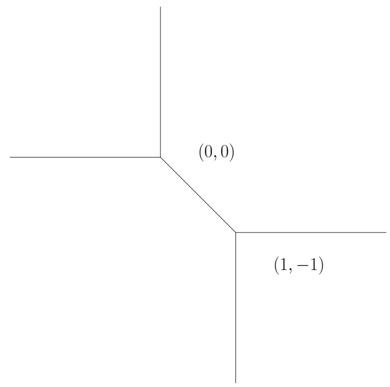

FIGURE 3.

$\operatorname{trop}(g) = \min(0, x, x+y, y+1)$, and

$$\begin{aligned}\operatorname{trop}(V(g)) = \{(w_1,w_2,w_3) : & w_2 = 0 \leq w_2+w_3, w_1+w_3 \text{ or}\\ & w_2+w_3 = 0 \leq w_2, w_1+w_2 \text{ or}\\ & w_1+w_3 = 0 \leq w_2, w_2+w_3 \text{ or}\\ & w_2 = w_2+w_3 \leq 0, w_1+w_3 \text{ or}\\ & w_2 = w_1+w_3 \leq 0, w_2+w_3 \text{ or}\\ & w_2+w_3 = w_1+w_3 \leq 0, w_2\}.\end{aligned}$$

For example $\{(w_1,w_2,w_3) : w_2 = 0 \leq w_2+w_3, w_1+w_3\} = \{(w_1,w_2,w_3) : w_2 = 0, w_3 \geq 0, w_1+w_3 \geq 0\} = \operatorname{pos}((-1,0,1),(1,0,0))$.

Then

$$\begin{aligned}\operatorname{trop}(V(g)) \cap \{w_1 = 1\} = & \{(1,0,w_3) : w_3 \geq 0\}\\ & \cup\{(1,w_2,-w_2) : 0 \leq w_2 \leq 1\}\\ & \cup\{(1,w_2,-1) : w_2 \geq 1\}\\ & \cup\{(1,w_2,0) : w_2 \leq 0\}\\ & \cup\emptyset\\ & \cup\{(1,1,w_3) : w_3+1 \leq 0\}.\end{aligned}$$

Labelling these cones A through E, these are shown in Figure 4.

These reductions mean that if $I \subset \overline{Q(t)}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ then we can compute $\operatorname{trop}(V(I))$ by doing computations in a Laurent polynomial ring with coefficients in $\mathbb{Q}$, so we need only discuss how to compute tropical varieties in this context.

A first algorithm to compute $\operatorname{trop}(V(I))$ in this context is to first homogenize the ideal I to get an ideal $I^h \in \mathbb{Q}[x_0, \ldots, x_n]$. We can then compute the Gröbner fan of I^h, and throw away those cones containing some w in their relative interior with $\operatorname{in}_w(I^h)$ containing a monomial. A problem with this, however, is that it is quite inefficient, as there can be many more cones in the Gröbner fan than there are in the tropical variety. In [BJS+07] an example is givien of a family of ideals in three

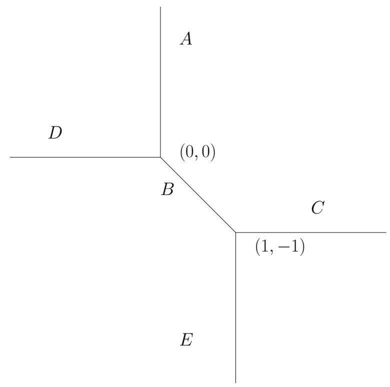

FIGURE 4.

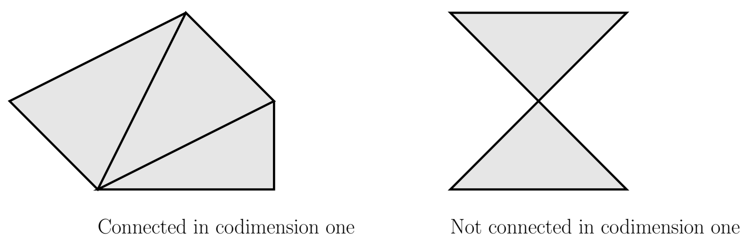

FIGURE 5.

variables indexed by $p \in \mathbb{N}$ where the tropical varieties all have four rays, but the Gröbner fan of the ideal I_p has at least $1/4(p+1)$ rays.

The key fact for the algorithm used by `gfan` is the following theorem.

Theorem 1. [BJS+07] *Let $X \subset T_K^n$, where* $\mathrm{char}(K) = 0$*, be an irreducible variety of dimension d. Then X is a pure polyhedral complex of dimension d that is connected in codimension one.*

Here "connected in codimension one" means that there is a path

$$\sigma = \sigma_1 \succ \tau_1 \prec \sigma_2 \succ \tau_2 \prec \sigma_3 \tau_3 \dots \tau_s \prec \sigma_s = \sigma'$$

connecting any two d-dimensional polyhedra σ, σ' of $\mathrm{trop}(X)$, where $\tau \prec \sigma$ means "τ is a facet ((d-1)-dimensional face) of σ. This is illustrated in Figure 5.

The algorithm used by `gfan` starts by computing a d-dimensional cone in the Gröbner fan of I^h for which the initial ideal does not contain a monomial. For each facet of this cone, it computes the neighbouring cones in the tropical variety, and then "walks" around the tropical variety until all cones have been visited. Theorem 1 guarantees that every cone will be found in this fashion. The computation of computing neighbouring cones can be effectively reduced to the case $d = 1$. For more details, see [BJS+07].

The program `gfan` [Jen], written by Anders Jensen, can be accessed via your `schwartz` accounts, or by using Andrew Hoefel's applet available at

`http://www.mathstat.dal.ca/~handrew/gfan/index.php`

To use it, create a file containing content like:

```
Q[x_1,x_2,x_3,x_4,x_5]
{x_1+x_2+x_3+x_4+x_5, x_1+2*x_2+3*x_3}
```

Note that unlike `M2` there is only one `Q`x in the name of the rational numbers. To get the starting cone, type `gfan_tropicalstartingcone < myfile >myfile.cone`, where `myfile` is the name of the file you created above, and you can change the name of `myfile.cone`. This will give output like:

```
Q[x_1,x_2,x_3,x_4,x_5]
{
x_2+2*x_3,
x_1-x_3}
{
x_2-x_5-x_4+2*x_3,
x_1+2*x_5+2*x_4-x_3}
```

This is a list of the initial ideal, and then the ideal. If you are using the web-applet, you will enter the input in the dialogue box, and the output will be on the screen below, which you will need to paste back into the dialogue box for the next step. Next type `gfan_tropicaltraverse <myfile.cone >myfile.output`. This will give output of the form:

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
5

DIM
3

LINEALITY_DIM
1

RAYS
-1 0 0 0 0 # 0
0 -1 0 0 0 # 1
-1 -1 -1 0 0 # 2
0 0 -1 0 0 # 3
1 1 1 1 0 # 4
0 0 0 -1 0 # 5

N_RAYS
6

LINEALITY_SPACE
1 1 1 1 1
```

```
ORTH_LINEALITY_SPACE
0 0 0 1 -1
0 0 1 0 -1
0 1 0 0 -1
1 0 0 0 -1

F_VECTOR
1 6 10

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}
{3}
{4}
{5}
{0 2} # Dimension 3
{2 3}
{1 2}
{0 4}
{1 4}
{0 5}
{1 5}
{4 5}
{3 4}
{3 5}

MAXIMAL_CONES
{0 2} # Dimension 3
{2 3}
{1 2}
{0 4}
{1 4}
{0 5}
{1 5}
{4 5}
{3 4}
{3 5}

PURE
1

MULTIPLICITIES
1 # Dimension 3
1
```

```
1
1
1
1
1
1
1
1
```

This says that the tropical variety has lineality space span($\mathbf{1}$), and is three-dimensional The quotient by the lineality space gives a fan with six rays, and ten two-dimensional cones, which are listed.

Note that this is not what we got when we did this example on Friday. Firstly this is because `gfan` uses the max convention instead of the min convention. Secondly, it is because gfan has amalgamated some of the cones we constructed. We had rays $\mathbf{e}_i$ for $1 \leq i \leq 5$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, and then $\mathbf{e}_i + \mathbf{e}_j$ for $1 \leq i < j \leq 5$ with $\{i, j\} \not\subset \{1, 2, 3\}$. Two of the cones were $\operatorname{span}(\mathbf{1}) + \operatorname{pos}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_4)$, and $\operatorname{span}(\mathbf{1}) + \operatorname{pos}(\mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_4)$. The union of these two cones is $\operatorname{span}(\mathbf{1}) + \operatorname{pos}(\mathbf{e}_1, \mathbf{e}_4)$.

Warning: The program `gfan` assumes that the ideal generated by your input polynomials is prime, so the variety is irreducible. It will not (necessarily) give the correct answer if your input variety is not irreducible.

Tropical Bases.

Definition 2. Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. A set $\{f_1, ;f_s\} \subset I$ is a *tropical basis* for I if

$$\operatorname{trop}(V(I)) = \bigcap_{i=1}^{s} \operatorname{trop}(V(f_i)).$$

Example: Let $I = \langle x + y + z, x + 2y \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. Then $\operatorname{trop}(V(I)) = \operatorname{span}(\mathbf{1})$. Any two polynomials in the set

$$\{x + 2y, y - z, x + 2z\}$$

form a tropical basis for I, as $\operatorname{trop}(V(x+2y)) = \{w \in \mathbb{R}^3 : w_1 = w_2\}$, $\operatorname{trop}(V(y-z)) = \{w \in \mathbb{R}^3 : w_2 = w_3\}$, and $\operatorname{trop}(V(x + 2z)) = \{w \in \mathbb{R}^3 : w_1 = w_3\}$. The set $\{x + y + z, x+2y\}$ is not a tropical basis, as the vector $(0, 0, 1) \in \operatorname{trop}(V(x+y+z)) \cap \operatorname{trop}(V(x+2y))$, but $(0, 0, 1) \notin \operatorname{trop}(V(I))$. Note that $\operatorname{trop}(V(x + y + z)) \cap \operatorname{trop}(V(x + 2y))$ is not a balanced polyhedral complex.

Example: Let $I = \langle f \rangle$ be a principal ideal. Then $\{f\}$ is a tropical basis for I. This was Q4 of Exercises 3.

Proposition 3. *Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then I has a finite tropical basis.*

Proof. Let I^h be the homogenization of I in $K[x_0, \dots, x_n]$, and let Σ be the Gröbner complex of I^h. There are only finitely many polyhedra $\sigma \in \Sigma$, and for each $\sigma \in \Sigma$ the initial ideal $\operatorname{in}_w(I^h)$ is constant for all w in the relative interior of σ, so we may denote it by $\operatorname{in}_\sigma(I^h)$. Given $\sigma \in \Sigma$ with $\operatorname{in}_w(I^h)$ containing a monomial for w in the relative interior of σ (so $\mathrm{in}_w(I) = \langle 1 \rangle$), we can find $\tilde{f}_\sigma \in I^h$ with $\mathrm{in}_w(\tilde{f}_\sigma)$ a monomial. Let $f_\sigma \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ denote the dehomogenization of $\tilde{f}_\sigma$, and let

$$\mathcal{G} = \{ f_\sigma : \mathrm{in}_\sigma(I^h) \text{ contains a monomial } \}.$$

Note that $\mathcal{G}$ is a finite set, and

$$\bigcap_{f_\sigma \in \mathcal{G}} \mathrm{trop}(V(f_\sigma)) \subseteq \mathrm{trop}(V(I)),$$

since $\mathrm{relint}(\sigma) \cap \mathrm{trop}(V(f_\sigma)) = \emptyset$. But since $f_\sigma \in I$, we have $\mathrm{trop}(V(I)) \subseteq \mathrm{trop}(V(f_\sigma))$, so

$$\bigcap_{f_\sigma \in \mathcal{G}} \mathrm{trop}(V(f_\sigma)) = \mathrm{trop}(V(I)).$$

$\square$

Open Problem: Give a good algorithm to compute a tropical basis.

There is an algorithm implicit in the above proof, but it involves knowing the Gröbner complex of I^h, which is hard to compute.

Definition 4. Suppose $\Bbbk$ includes into K. Let $I \subset \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a linear ideal (generated by linear polynomials). A linear polynomial $f = \sum_{i=1}^n a_i x_i$ is a *circuit* for I if $\{i : a_i \neq 0\}$ is minimal with respect to inclusion. This means that there is no $g = \sum_{j=1}^n b_j x_j \in I$ with $\{j : b_j \neq 0\} \subsetneq \{i : a_i \neq 0\}$. Note that there are only a finite number of circuits up to scaling.

Exercise: Let I be a linear ideal. Then the set of circuits form a tropical basis for I.

Remark 5. There is also a stricter notion of tropical basis, where we require that the tropical basis in addition be a Gröbner basis for I with respect to any $w \in \mathrm{trop}(V(I))$. This is also finite.

References

[BJS$^+$07] T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, and R. R. Thomas, *Computing tropical varieties*, J. Symbolic Comput. **42** (2007), no. 1-2, 54–73. MR **2284285 (2007j:**14103)

[Jen] Anders N. Jensen, *Gfan, a software system for Gröbner fans.* Available at `http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html`.