AARMS TROPICAL GEOMETRY - LECTURE 1

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These notes mostly represent what I intend to do in class, rather than broader background. Please let me know (D.Maclagan at warwick.ac.uk) if you find any typos, no matter how small. Since these are just lecture notes, no claim is made for completeness or full references.

What is tropical geometry?

First answer: (Algebraic) geometry where instead of working over the complex numbers (or some other field) we work over the *tropical semiring* $(\mathbb{R}, \oplus, \otimes)$. Here $\oplus$ is the usual minimum, and $\otimes$ is the usual addition.

Example: $(5 \oplus 6) \otimes 7 = 12$.

When necessary we consider $(\mathbb{R} \cup \infty, \oplus, \otimes)$, so ∞ becomes the additive identity. Then $(\mathbb{R} \cup \infty, \oplus, \otimes)$ is a semiring (ie associative, distributative etc - just no additative inverse).

In algebraic geometry we often work with polynomials. In tropical geometry we "tropicalize" these polynomials, which turns them into piecewise linear functions.

Example: $f(x, y) = x^2 + y^2 - 1$. This tropicalizes to $\text{trop}(f) = x^2 \oplus y^2 \oplus 0 = \min(2x, 2y, 0)$. This is a piecewise linear function. (See below for why the -1 turns into 0). See Figure 1.

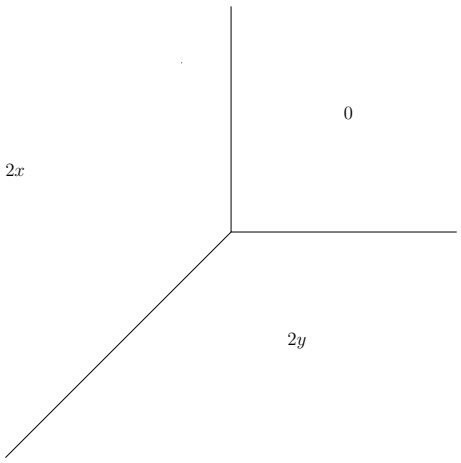

FIGURE 1.

In algebraic geometry we study the common zeros of polynomial equations (Warning: Oversimplification!). These are called varieties. Tropically this corresponds to taking the nonlinear locus of the polynomial $\text{trop}(f)$.

Example: Let $f(x, y) = x + y + 1$. The variety of f is the set $\{(x, y) \in \mathbb{C}^2 : x + y + 1 = 0\}$, which is a line in $\mathbb{C}^2$. Then $\text{trop}(f) = \min(x, y, 0)$. This is a piecewise

linear function with graph shown in Figure 2. The nonlinear locus is the three line segments $x = y \leq 0$, $x = 0 \leq y$, and $y = 0 \leq x$. This is also shown in Figure 2.

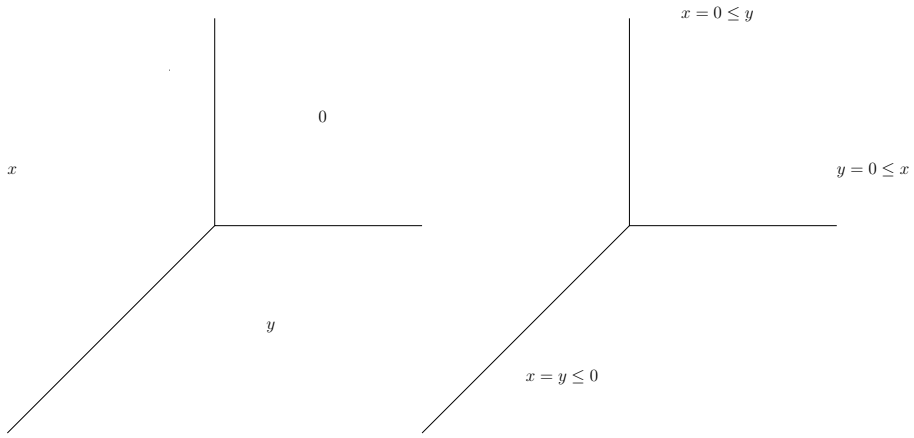

FIGURE 2.

Thus varieties turn into polyhedral complexes under the tropicalization map.

Warning: An issue with this first answer is that not everything tropicalizes well. In particular, maps between varieties do not tropicalize precisely as expected. ("Tropicalization is not functorial"). For this reason we will be careful over the next two weeks to define things formally.

Motivation

Why tropicalize? A first reason is that polyhedral geometry is (often) easier than algebraic geometry. Many invariants of the variety become invariants of the resulting polyhedral complex.

Example: Let $f = x + y + 1$. Then the set $f = 0$ is the line $\{(t, -1 - t) : t \in \mathbb{C}\}$ in $\mathbb{C}^2$, so is one-dimensional. The tropical variety, shown on the right in Figure 2, is also one-dimensional.

It is true in general (we will see later) that dimension is preserved under tropicalization. Other (primarily intersection theoretic) invariants are also preserved.

Motivating Example: Counting Curves

One of the first successful applications of tropical geometry has been to enumerative geometry, primarily in the work of Mikhalkin. This allows a simple answer to the classical question of counting the number of rational curves in $\mathbb{P}^2$ of a given degree d passing through a set of fixed points in general position. A curve C in $\mathbb{P}^2$ is given by a homogeneous polynomial $f(x, y, z) = 0$. The curve C is rational if it is isomorphic to $\mathbb{P}^1$ (informally, if there is a parameterization $\phi : \mathbb{C} \to C$ so C is the Zariski closure of the image of ϕ). The degree of the curve is the degree of the polynomial. In order for this number to be finite, we ask that the points be in general position, and that there be $3d-1$ of them. Here "general position" means that there is a (Zariski) open set in $(\mathbb{P}^2)^{3d-1}/S_{3d-1}$ for which this number is constant.

Definition 1. Let $N_{0,d}$ be the number of rational curves of degree d passing through a collection of $3d - 1$ points in $\mathbb{P}^2$ in general position.

Example:

$d = 1$ A curve of degree one is a straight line, which is rational. Thus $N_{0,1}$ is one, as there is a unique line joining any two distinct points in $\mathbb{P}^2$.
$d = 2$ All curves of degree two are rational, and there is a unique curve through any five points in general position in $\mathbb{P}^2$ (see exercises). Thus $N_{0,2} = 1$.
$d = 3$ $N_{0,3} = 12$. This was computed by Steiner in 1848, and was possibly known earlier.
$d = 4$ $N_{0,4} = 620$. This was computed by Zeuthen in 1873.
$d = 5$ $N_{0,5} = 87304$. This (and all later ones) were unknown until the early 90s.
$d = 6$ $N_{0,6} = 26312976$.

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In 1994 Kontsevich gave a recursive formula that determines all of these numbers from $N_{0,1} = 1$. This involved developing the moduli space of stable maps, which is at the foundations of Gromov-Witten theory. Giving a self-contained proof of this would take more than this entire course. However in the last week we will (hopefully) give a self-contained proof of the Kontsevich recursion using tropical methods.

We now return to the question of tropicalizing polynomials. Earlier we said $\mathrm{trop}(x^2 + y^2 - 1) = \min(2x, 2y, 0)$. The -1 turned mysteriously into a 0. We will now partially explain this (though a full explanation will be later in the week).

Let

$$K = \mathbb{C}\{\{t\}\} = \cup_{n\geq 1}\mathbb{C}((t^{1/n},)),$$

where by $\mathbb{C}((t^{1/n}))$ we mean the ring of Laurent series in the variable $t^{1/n}$. This ring K is the ring of *Puiseux series*. An element $a \in K$ has the form

$$a = \sum_{q\in\mathbb{Q}} a_q t^q,$$

where $\{q \in \mathbb{Q} : a_q \neq 0\}$ is bounded below and has a common denominator.

Write $K^* = K \setminus \{0\}$. Let $\mathrm{val} : K^* \to \mathbb{R}$ be given by $\mathrm{val}(a) = \min\{q : a_q \neq 0\}$. This lets us define the tropicalization of a polynomial formally.

Definition 2. Let $S := \mathbb{C}[x_1, \ldots, x_n]$, and write

$$f = \sum_{u\in\mathbb{N}^n} c_u x^u,$$

where $x^u := \prod_{i=1}^n x_i^{u_i}$. Then

$$\mathrm{trop}(f) = \min_{u\in\mathbb{N}^n : c_u\neq 0}(\mathrm{val}(c_u) + \sum_{i=1}^n u_i x_i).$$

The tropical hypersurface of f is

$$\mathrm{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{ the minimum in the definition of } \mathrm{trop}(f)(w) \text{ is achieved twice}\}.$$

Note that we could also define $\mathrm{val}\colon K \to \mathbb{R} \cup \infty$ by setting $\mathrm{val}(0) = \infty$. Then $\mathrm{trop}(f) = \min_{u\in\mathbb{N}^n} \mathrm{val}(c_u) + \sum_{i=1}^n u_i x_i)$. Note also that $\mathrm{val}(-1) = 0$, so this explains the earlier zero.

Example: Let $f = tx^2 + 2xy + 3ty^2 + 5x + 7y - (t^2 + t^5)$. Then $\mathrm{trop}(f) = \min(2x + 1, x + y, 2y + 1, x, y, 2)$. This function is illustrated in Figure 3.

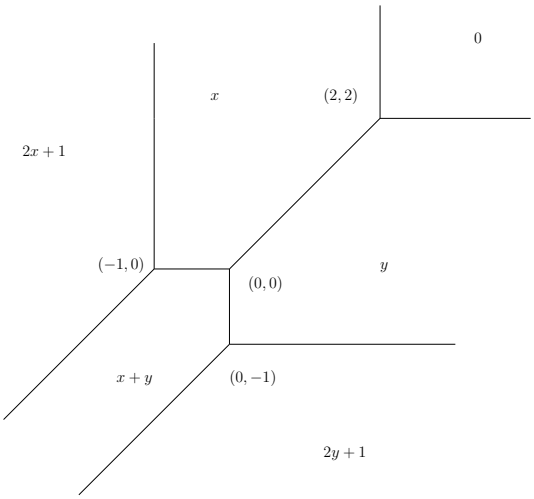

FIGURE 3.

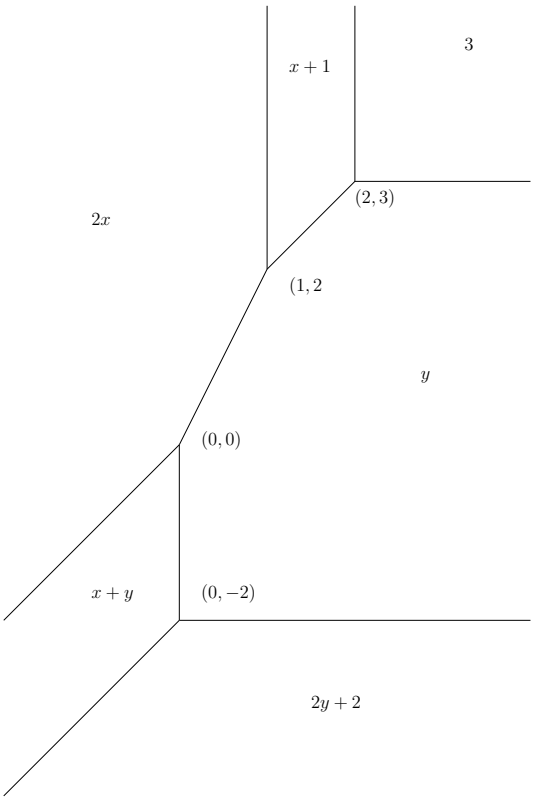

FIGURE 4.

Example: Let $f = (t^2 - t^{5/2})y^2 + 5x^2 - 7xy + 8y - tx + t^3$. Then $\mathrm{trop}(f) = \min(2 + 2y, 2x, x + y, y, x + 1, 3)$. This is illustrated in Figure 4.
Example: Let $f = tx^2 + 3xy - 7(t^3 + t^5)y^2 + ty - 7x + 5$. Then $\mathrm{trop}(f) = \min(2x + 1, x + y, 2y + 3, y + 1, x, 0)$. This is illustrated in Figure 5.

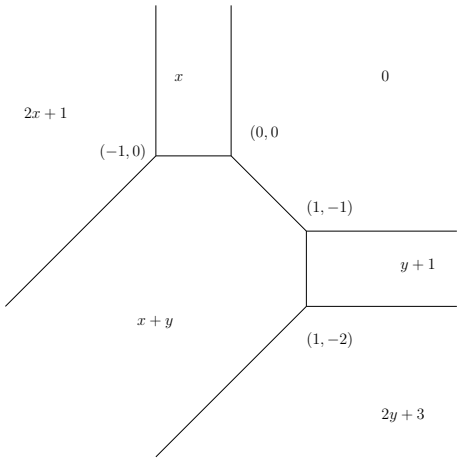

FIGURE 5.

Example: Let $f = t^2x - 7(t+t^3)y + t^5$. Then $\text{trop}(f) = \min(x+2, y+1, 5)$. This is illustrated in Figure 6.

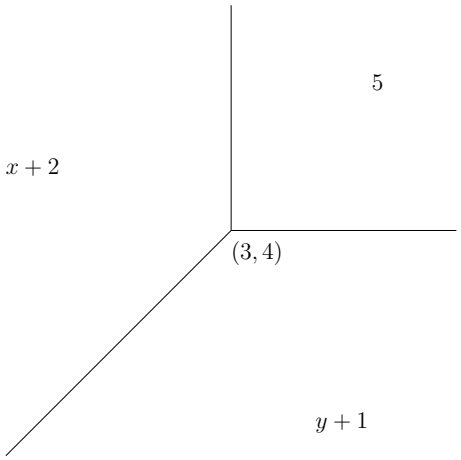

FIGURE 6.

Example: Let $f = t^3x^2 - 7tx + 8xy - 7y^2 + 6$. Then $\text{trop}(f) = \min(2x+3, x+1, x+y, 2y, 0)$. This is illustrated in Figure 7.
Example: Let $f = t^4x^3 + t^2x^2y + tx^2y^2 + t^4y^3 + tx^2 + xy + ty^2 + x + y + t$. Then $\text{trop}(f) = \min(2x+4, 2x+y+2, x+2y+2, 3y+4, 2x+1, x+y, 2y+1, x, y, 1)$. This is illustrated in Figure 8.
Example: Let $f = x^2+2xy+3y^2+4x+5y+6$. Then $\text{trop}(f) = \min(2x, 2y, x, y, 0)$. This is illustrated in Figure 9.

Note two important aspects of these pictures: Firstly, in almost all cases, there are the same number of "tentacles" in each of three directions, and that number is the degree of the polynomial. Secondly, in almost all cases, at each vertex of the graph, the sum of the three vectors emanating from that vertex add to zero. In fact, with the appropriate notion of multiplicity (see next week), both of these are true in all cases.

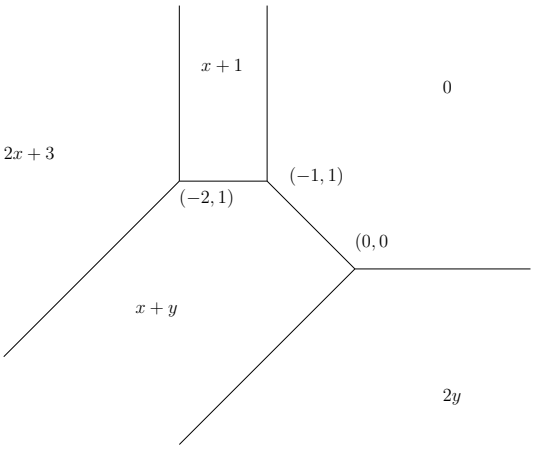

FIGURE 7.

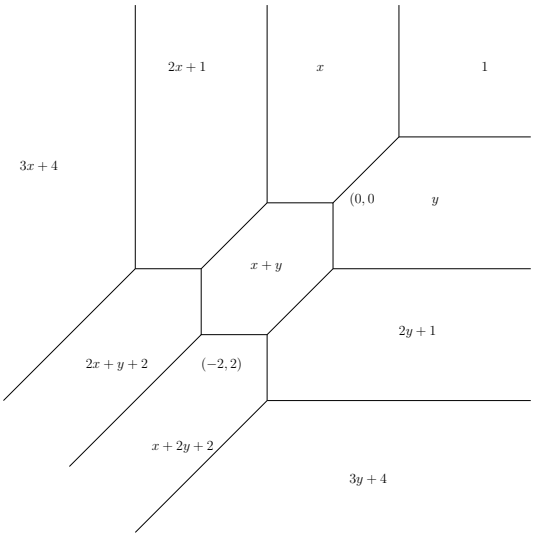

FIGURE 8.

Outline of Course:

Week 1: Introduction to varieties. Background tools, such as valuations and Gröbner bases.

Week 2: Fundamental theorems on tropical varieties. Some basic examples.

Week 3: More examples. Connections to toric varieties.

Week 4: Enumerative geometry. Presentations.

Presentations: During this month you will read a research paper or two in small groups (3–5) and do a presentation on the material during the last few class periods. There is a list of possible papers on the webpage. Check that out today!!

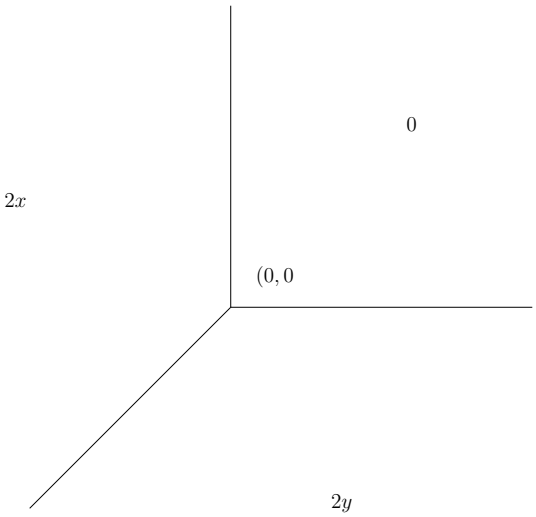

FIGURE 9.