AARMS TROPICAL GEOMETRY - EXERCISES 5

DIANE MACLAGAN

(1) Let I be an ideal in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by linear forms, and let $\mathcal{C}$ be the set of *circuits* of I. Recall that a circuit is a polynomial $f = \sum_{i=1}^n a_i x_i \in I$ with support $\{i : a_i \neq 0\}$ minimal with respect to inclusion. Show that $\mathcal{C}$ is a tropical basis for I.

(2) Let τ be a trivalent tree with n leaves. Show that τ has $2n-3$ edges.

(3) Let $\phi : T^n \to T^m$ be a morphism of tori, given by $\phi(t)_i = t^{\mathbf{u}_i}$, and let U be the $n \times m$ matrix with columns $\mathbf{u}_1, \dots, \mathbf{u}_m$. Show that ϕ is surjective if and only if U has rank m.

(4) Let $X = V(x+y+z+1) \subseteq T^3$. For each of the following maps of tori $\phi\colon T^3 \to T^m$ compute $\overline{\phi(X)} \subset T^m$, and verify that $\mathrm{trop}(\overline{\phi(X)}) = \{U^T w : w \in \mathrm{trop}(X)\}$, where U is the $3 \times m$ matrix with columns $\mathbf{u}_1, \dots, \mathbf{u}_m$ for $\phi(t)_i = t^{\mathbf{u}_i}$.

   (a) $\phi : T^3 \to T^2$ given by $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   (b) $\phi : T^3 \to T^2$ given by $U = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$.

   (c) $\phi : T^3 \to T^4$ given by $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(5) For which of the following $X \subset T^3$ is $0 \in \overline{X} \subset \mathbb{A}^3$? Compute $\mathrm{trop}(X)$ and verify the statement of the Theorem.

   (a) $X = V(3x^2 + 3xy + 2x + y)$;

   (b) $X = V(3x^2 + 3xy + 2x + y + 1)$;

   (c) $X = V(x+y+z, x+2y)$;

   (d) $X = V(x+y+z+1, x+2y+3z)$;

(6) (a) Check that $T^2 \subset \mathbb{P}^2$ is a Zariski-dense subset, and that there is an action of T^2 on $\mathbb{P}^2$ that extends the action of T^2 on itself.

   (b) List the orbits of T^2 in $\mathbb{P}^2$.

   (c) Let $X = V(x + y + x^2y + xy^2) \subset T^2$, and let $\overline{X}$ be the closure of X in $\mathbb{P}^2$. How does $\overline{X}$ intersect each of the T^2-orbits of $\mathbb{P}^2$?

   (d) Draw $\mathrm{trop}(X) \subset \mathbb{R}^2$. How does this relate to your previous answer?