# AARMS TROPICAL GEOMETRY - EXERCISES 4

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These questions cover Wednesday of week two to Tuesday of week three.

(1) Show that the definition of the star of a polyhedron $\sigma$ in a polyhedral complex $\Sigma$ does not depend on the choice of the point $w \in \sigma$.

(2) For each of the following polynomials $f$
  (a) Compute $\operatorname{trop}(V(f))$.
  (b) For each polyhedron $\sigma$ in the polyhedral complex $\operatorname{trop}(V(f))$ compute $\operatorname{in}_w(I)$ for any $w$ in the relative interior of $\sigma$, and compute directly $\operatorname{trop}(V(\operatorname{in}_w(I))$. Check that this is the appropriate star.
  (c) Let $A$ be an $r \times n$ matrix of rank $r$ with entries in $\mathbb{k}$, and let $B$ be a $(n-r) \times n$ matrix whose rows form a basis for $\ker(A)$. Show that if the first $(n-r) \times (n-r)$ submatrix of $B$ is invertible, then the last $r \times r$ submatrix is also invertible.
  (d) Check that $\operatorname{trop}(V(f))$ is a weighted balanced polyhedral cone. (The polynomials have been chosen so that all weights are one).

  (I do not expect to see all of these written up completely - do enough until you are satisfied you understand what is going on).
  (a) $f = x + ty + t^3 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$;
  (b) $f = x^2 + txy + x + y + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$;
  (c) $f = x + y + x^2y + xy^2 + x^2y^2 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$;
  (d) $f = x + y + z + 1 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

(3) For each of the following linear varieties compute $\operatorname{trop}(X)$. Repeat the verifications of parts (b) and (c) of the previous question in this context.
  (a) $X = V(x_1 + x_2 + x_3 + x_4, x_1 + 2x_2 + 4x_3 - x_4) \subset T^4$;
  (b) $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_1 - x_2 + 3x_3 + 4x_4 + 7x_5) \subset T^5$;
  (c) $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + 3x_4 - x_5) \subset T^5$.

(4) Let $X \subset T^n$ be defined by linear equations with coefficients in $\mathbb{k}$.. Show that the multiplicity of each cone in $\operatorname{trop}(X)$ is one.

(5) The goal of this exercise is to explain in detail how to compute $\operatorname{in}_w(I)$ using a computer algebra package such as `Macaulay 2`.
  (a) Read the proof of Lemma 8 and Corollary 9 of Lecture 8.
  (b) Given $I = \langle f_1, \dots, f_r \rangle \subseteq \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\tilde{f}_i$ be the polynomial obtained from $f_i$ by clearing denominators. Check that $I = \langle \tilde{f}_1, \dots, \tilde{f}_r \rangle \subset \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.
  (c) Let $\bar{I} = (\langle \tilde{f}_1, \dots, \tilde{f}_r \rangle : (\prod_{i=1}^n x_i)^\infty) \subset \mathbb{k}[x_1, \dots, x_n]$. Show that $\bar{I} = I \cap \mathbb{k}[x_1, \dots, x_n]$.
  (d) Let $I = \langle x^2 - y, y^2 - x \rangle \subset K[x^{\pm 1}, y^{\pm 1}]$. Use $I$ to show that the saturation step is necessary in the previous exercise.

(e) Let $f'_i$ be the homogenization of $\tilde{f}_i$ using the variable $x_0$. Let $J = (\langle f'_1, \dots, f'_r \rangle : x_0^\infty) \subseteq \Bbbk[x_0, \dots, x_n]$. Show that $V(J) \subset \mathbb{P}^n$ is the closure of the image under the map $x \mapsto (1 : x)$ of $V(\bar{I}) \subset \mathbb{A}^n$. (The saturation step is again necessary here: an optional exercise is to find an example showing this).

(f) Show that for most choices of $v \in \mathbb{R}^n$ we have that $\mathrm{in}_v(\mathrm{in}_w(J))$ is a monomial ideal.

(g) Use the previous parts of this exercise to describe how to use a computer algebra package to compute $\mathrm{in}_w(I)$.

(h) In `Macaulay 2` we describe a ring using
`R=QQ[x_0..x_5, Weights=>{1,2,3,4,5,6}].`
In this example $\tilde{w} = (1, 2, 3, 4, 5, 6)$. If $f = \sum a_u x^u$, let $\mathrm{in}_{\tilde{w}}(f) = \sum_{\tilde{w}\cdot u \text{ is maximal}} a_u x^u$. (Note the difference between min and max here!) Show that $\mathrm{in}_{\tilde{w}}(f) = \mathrm{in}_{\tilde{w}+\lambda\mathbf{1}}(f)$, where $\mathbf{1} \in \mathbb{R}^{n+1}$ is the all-ones vector, and $\lambda \in \mathbb{R}$.
The command `leadTerm(1,J)` computes generators for the ideal $\langle \mathrm{in}_{\tilde{w}}(f) : f \in J \rangle$. Weights must be positive in `Macaulay 2`. Conclude that if $w \in \mathbb{R}^n$, setting `Weights=>w2` where $w_2 = N\mathbf{1} - (0, w)$ for $N \gg 0$ lets `leadTerm(1,J)` compute $\mathrm{in}_w(I)$.

(6) The goal of this exercise is to prove that $\mathrm{trop}(X)$ is a weighted balanced complex when $X$ is a curve in $T^2$. Since the balanced condition is a local condition, we only need to consider ideals in $\Bbbk[x_1, \dots, x_n]$, so $\mathrm{trop}(X)$ is a one-dimensional fan. Let

$$f = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j,$$

Let

$$P = \mathrm{conv}((i, j) : a_{ij} \neq 0) \subset \mathbb{R}^2.$$

where $a_{ij} \in \Bbbk$. Then *inner normal* to an edge $e$ of $P$ is a vector $w \in \mathbb{R}^2$ with $w \cdot (i, j) < w \cdot (i', j')$ if $(i, j) \in e$ and $(i', j') \in P \setminus e$.

(a) Let $f = x + y + x^2y + xy^2 + x^2y^2$. Draw $P$, and the inner normal to each edge of $P$. Check that this picture is $\mathrm{trop}(V(f))$.

Note that we can always choose the inner normal $w$ to an edge $e$ to be in $\mathbb{Z}^2$. A vector $w \in \mathbb{Z}^2$ is *primitive* if $\gcd(w_1, w_2) = 1$.

(b) Show that every edge $e \in P$ has a unique primitive inner normal $w_e \in \mathbb{Z}^2$.

(c) Show that for any $f \in \Bbbk[x^{\pm 1}, y^{\pm 1}]$ the tropical variety $\mathrm{trop}(V(f))$ is equal to $\cup_e \mathrm{pos}(w_e)$, where the union is over all edges $e$ of $P$.

(d) Let $w_e$ be the primitive inner normal to an edge $w_e$ of $P$. Show that there is a monomial $m = x^a y^b$ and a polynomial $g \in \Bbbk[m]$ for which

$$V(\mathrm{in}_{w_e}(f) = V(g)$$

and $\deg(g)$ is the *lattice length* of the edge $e$ (one less than the number of lattice points in $e$). Thus conclude that the multiplicity $m_e$ of the ray $\mathrm{pos}(w_e)$ of $\mathrm{trop}(V(f))$ is the lattice length of $e$.

(e) Verify the previous question for $f = x + y + x^2y + xy^2 + x^2y^2$.

(f) Conclude that $\sum_e m_e w_e = 0$, so $\text{trop}(V(f))$ is a balanced weighted polyhedral fan. Hint: Let $\mathbf{e}$ be the vector corresponding to the edge $e$, oriented clockwise. Note that $\sum_e \mathbf{e} = \mathbf{0}$.

(7) Let $f = tx^2 + xy + ty^2 + x + y + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Compute $\text{trop}(V(f))$, and compare with $\text{trop}(V(g)) \cap \{w_1 = 1\}$ for $g = tx^2 + xy + ty^2 + x + y + t \in \mathbb{C}[t^{\pm 1}, x^{\pm 1}, y^{\pm 1}]$.

(8) Play with `gfan`. Plug in some of the examples we have computed in lectures and from the notes and check that `gfan` agrees. Warning: Remember that `gfan` uses the max convention instead of our min convention.

(9) Use `gfan` to compute the tropical variety of $X = V(I) \subset T^{10}$, where $I = \langle x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}, x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}, x_{12}x_{45} - x_{14}x_{35} + x_{15}x_{24}, x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34}, x_{23}x_{45} - x_{35}x_{24} + x_{25}x_{34}\rangle \subset \Bbbk[x_{12}^{\pm 1}, x_{13}^{\pm 1}, x_{14}^{\pm 1}, x_{15}^{\pm 1}, x_{23}^{\pm 1}, x_{24}^{\pm 1}, x_{25}^{\pm 1}, x_{34}^{\pm 1}, x_{35}^{\pm 1}, x_{45}^{\pm 1}]$. Also compute the tropical variety of $X = V(J) \subset T^6$ where $J = \langle 1 - y_1 + y_3, 1 - y_2 + y_4, y_1 - y_2 + y_5\rangle \subset \Bbbk[y_1^{\pm 1}, \dots, y_5^{\pm 1}]$ (Hint: you will need to homogenize first). Compare your answers. Can you explain what you observe?

(10) Let $X = V(xy + y^2 + 2xz + 2yz - xw - yw, x^2 - y^2 - 3xz - 3yz + 3xw + 3yw) \subset T^4$. Show that $X$ is not irreducible. Hint: The `decompose` and `intersect` commands in M2 may help. What does `gfan` think is the tropical variety of $X$? Is this correct?

(11) (Only for the very computationally inclined). Look at the paper `math.AG/0507563` for details of the algorithm `gfan` uses to compute tropical varieties.

(12) Describe the tropical variety of the Grassmannian $G(2, 6)$. How many cones of each dimension are there?