AARMS TROPICAL GEOMETRY - EXERCISES 3

DIANE MACLAGAN

(1) Let K be an algebraically closed field with a nontrivial valuation $\mathrm{val} : K \to \mathbb{R} \cup \infty$. Show that im(val) is dense in $\mathbb{R}$.
(2) Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Show that $\mathrm{in}_w(fg) = \mathrm{in}_w(f)\,\mathrm{in}_w(g)$.
(3) Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Show that if $g \in \mathrm{in}_w(I)$ then $g = \mathrm{in}_w(f)$ for some $f \in I$.
(4) Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $I = \langle f \rangle$. Show that $\mathrm{trop}(V(f)) = \cap_{g \in I} \mathrm{trop}(V(g))$. This can be rephrased as "hypersurfaces tropicalize to hypersurfaces".
(5) Let $f = tx_1^2 + x_1x_2 + tx_2^2 + x_0x_1 + x_0x_2 + t^4x_0^2 \in \mathbb{C}\{\{t\}\}[x_0, x_1, x_2]$. Compute the Gröbner complex of $I = \langle f \rangle$ (ie compute the polyhedra on which $\mathrm{in}_w(f)$ is constant). (Part of this exercise is taking the common generalization of Gröbner bases in $\mathbb{C}[x_0, x_1, x_2]$ and those in $\mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$). Use your answer to draw the tropical variety of $V(f) \subseteq \mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$.
(6) Verify (as much as possible) the fundamental theorem of tropical geometry for $X = V(f)$ for the following polynomials $f \in \mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$:
   (a) $f = 3x_1 + t^2x_2 + 2t$;
   (b) $f = tx_1^2 + x_1x_2 + tx_2^2 + x_1 + x_2 + t$;
   (c) $f = x_1^3 + x_2^3 + 1$.
(7) Let $S = K[x_1^{\pm 1}, \dots, x_4^{\pm 1}]$. Describe $\mathrm{trop}(X)$ for the following subvarieties of T^4. Hint: Both are two-dimensional, so you could draw the graph of $\mathrm{trop}(X) \cap S^3$.
   (a) $X = V(x_1 + x_2 + x_3 + x_4 + 1, x_2 + 2x_3 + 3x_4 + 4)$;
   (b) $X = V(x_1 + x_2 + x_3 + x_4 + 1, x_2 + x_3 + 2x_4 + 2)$.
(8) Let $\phi : T^2 \to T^4$ be given by
$$\phi(t_1, t_2) = (t_1, t_1t_2, t_1t_2^2, t_1t_2^3).$$
   Let $X = \mathrm{im}(\phi)$. Compute $\mathrm{trop}(X)$.