AARMS TROPICAL GEOMETRY - EXERCISES 2

DIANE MACLAGAN

Valuations

(1) Show that the residue field of $\Bbbk\{\{t\}\}$ is isomorphic to $\Bbbk$.
(2) Let $K = \mathbb{Q}$ with the p-adic valuation. Show that the residue field of K is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.
(3) Show that if K is an algebraically closed field with a valuation $\mathrm{val} : K^* \to \mathbb{R}$, and $\Bbbk = R/\mathfrak{m}$ is its residue field, then $\Bbbk$ is algebraically closed. Give an example to show that if $\Bbbk$ is algebraically closed it does not automatically follow that K is algebraically closed.
(4) In the proof that $\Bbbk\{\{t\}\}$ is algebraically closed, explain why f_i has degree k_i and has a nonzero constant term.
(5) Apply the algorithm implicit in the proof that $\mathbb{C}\{\{t\}\}$ is algebraically closed to compute (the start of) a solution to the equation $x^2 + t + 1 = 0$. Check your answer with a computer algebra package (eg `puiseux` in `maple`).

Gröbner bases

(1) Let $I = \langle f \rangle \subset \Bbbk[x_0, \dots, x_n]$ be a principal ideal. Show that f is a Gröbner basis for I.
(2) Compute all the initial ideals $\mathrm{in}_w(f)$ of $f = 7x_0^2 + 8x_0x_1 - x_1^2 + x_0x_2 + 3x_2^2$ as w varies in $\mathbb{R}^2$. Draw the Gröbner fan of $\langle f \rangle$. (Hint: start by choosing some particular values of w).
(3) Show that if $\mathrm{in}_w(I)$ is a monomial ideal for $I \subset S = \Bbbk[x_0, \dots, x_n]$ then the monomials not in $\mathrm{in}_w(I)$ form a $\Bbbk$-basis for S/I.
(4) In this question you will compute the Gröbner fan of a principal ideal. The *Newton polytope* of a polynomial $f = \sum_{u \in \mathbb{N}^{n+1}} c_u x^u \in \Bbbk[x_0, \dots, x_n]$ is the convex hull in $\mathbb{R}^{n+1}$ of the exponents $\{u : c_u \neq 0\}$.
   (a) Draw the Newton polytope of $x_0^2 + x_0x_1 + x_1^2 + x_2^2$.
   If P is a polytope in $\mathbb{R}^n$, a point $\mathbf{v} \in P$ is a *vertex* if there is $\mathbf{w} \in \mathbb{R}^n$ for which $\mathbf{w} \cdot \mathbf{v} < \mathbf{w} \cdot x$ for all $x \in P \setminus \mathbf{v}$. The *normal cone* to P at $\mathbf{v}$ is the closure of the set of all such $\mathbf{w}$.
   (b) Let $P = \mathrm{conv}((0,0),(2,0),(0,2),(1,1),(2,2))$. What are the vertices of P? Draw the normal cone to each.
   The *normal fan* of P is the union of the normal cones to vertices of P. It is a polyhedral fan.
   (c) Draw the normal fan to the P of the previous question.
   (d) Show that the Gröbner fan (as we have defined it) of $\langle f \rangle$ is the $x_0 = 0$ slice of the normal fan to the Newton polytope of f.
(5) (For people who already knew something about Gröbner bases). It is more standard to define an initial ideal using a term order on the polynomial ring.

(a) Let $f = x_0^2 + x_0x_1 + x_1^2 + x_2^2$. For each lexicographic or degree reverse lexicographic term order $\prec$ find $w \in \mathbb{R}^2$ with $\text{in}_w(f) = \text{in}_\prec(f)$.

(b) In fact every term order can be represented by a vector w. You can read a proof, for example, in Proposition 2.4.4 of the notes available at `www.warwick.ac.uk/staff/D.Maclagan/papers/indialectures.pdf.gz` . See elsewhere in that chapter for hints on how to compute $\text{in}_w(I)$ using your favourite computer algebra package.

(6) Let $f = t^2x + 3ty + t^4 \in K[x^{\pm 1}, y^{\pm 1}]$, where $K = \mathbb{C}\{\{t\}\}$. Compute $\text{in}_w(f)$ for $w = (2, 5)$, and $w = (1, 2)$.

(7) Let $f = x + y + 1$. Draw $\{w \in \mathbb{R}^2 : \text{in}_w(f) \neq \langle 1 \rangle\}$. Repeat this with $f = tx^2 + xy + ty^2 + x + y + t$. Compare your pictures with $\text{trop}(V(f))$ in each case.

(8) Fix $I \subset \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $\bar{I} = I \cap \Bbbk[x_1, \ldots, x_n]$, and let $\tilde{J} = \langle \tilde{f} : f \in \bar{I} \rangle \subset \Bbbk[x_0, \ldots, x_n]$, where $\tilde{f}$ is the homogeneization of f using the variable x_0. Show that

$$\text{in}_w(J)|_{x_0=1} = \text{in}_w(I).$$

Optional extra: repeat with K.

(9) (Open ended for the more computationally minded:) Play with the software `gfan` (freely available from `http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html`).

(10) (Less open ended). If you don't download gfan, find someone else in the class who has.