AARMS TROPICAL GEOMETRY - EXERCISES 1

DIANE MACLAGAN

These questions cover approximately Monday - Wednesday of week one. You do not need to do every question! This week some of you may have seen some of the content before, so concentrate on the new material. Do at least one question from each days material - ask me for advice on which questions are most appropriate for your background if you're not sure. You are *strongly* encouraged to work together. I will ask you to create a solution set as a group. This will involve typing up the answer to approximately one question each a week.

Tropical Questions

(1) Check that $(\mathbb{R}, \oplus, \otimes)$ is a semiring.

(2) Draw a picture of the tropical curve corresponding to the following polynomials in $K[x, y]$:
  (a) $f = t^3x + (t + 3t^2 + 5t^4)y + t^{-2}$;
  (b) $f = (t^{-1} + 1)x + (t^2 - 3t^3)y + 5t^4$;
  (c) $f = t^3x^2 + xy + ty^2 + tx + y + 1$;
  (d) $f = 4t^4x^2 + (3t + t^3)xy + (5 + t)y^2 + 7x + (-1 + t^3)y + 4t$;
  (e) $f = tx^2 + 4xy - 7y^2 + 8$;
  (f) $f = t^6x^3 + x^2y + xy^2 + t^6y^3 + t^3x^2 + t^{-1}xy + t^3y^2 + tx + ty + 1$.

(3) The goal of this exercise is to show the connection between tropical curves in the plane and triangulations of a certain point configuration. It requires some basic knowledge of polyhedral geometry (and is probably the hardest exercise in this problem set). Ask for hints/help once you've thought about it a little.

Fix $d > 0$. Let $\mathcal{A}_d = \{(a, b) : a + b \leq d, a, b \geq 0\}$. Fix a polynomial $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^ay^b$ with $c_{ab} \in \mathbb{C}\{\{t\}\}$. The *regular triangulation* of $\mathcal{A}_d$ induced by f is obtained by taking the convex hull of the points $\{(a, b, \text{val}(c_{ab})) : (a, b) \in \mathcal{A}\}$ and taking the (projections of the) set of *lower faces*. These are the faces that you can see if you look from $(0, 0, -N)$ for $N \gg 0$.

Example: Let $d = 2$, so $\mathcal{A}_2 = \{(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)\}$. Let $f = tx^2 + xy + ty^2 + x + y + t^6$. We form the convex hull of the points $\{(2, 0, 1), (1, 1, 0), (0, 2, 1), (1, 0, 0), (0, 1, 0), (0, 0, 6)\}$. The lower faces of this polytope are illustrated in Figure 1.

  (a) Draw the regular triangulation of $\mathcal{A}_2$ corresponding to the polynomial $f = tx^2 + xy + t^3y^2 + x + ty + 1$.
  (b) Draw the regular triangulation of $\mathcal{A}_1$ corresponding to the polynomial $f = t^5x + t^3y + t^10$.
  (c) Draw the regular triangulation of $\mathcal{A}_3$ corresponding to the polynomial $f = t^3x^3 + tx^2y + txy^2 + t^3y^3 + tx^2 + xy + ty^2 + tx + ty + t^3$.

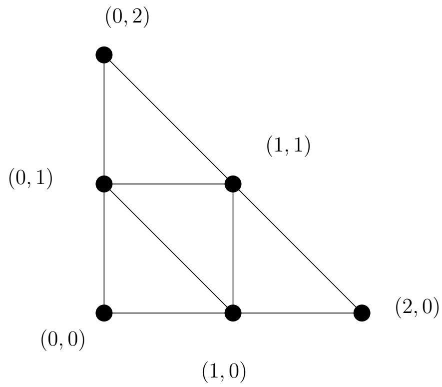

FIGURE 1.

The dual graph to a triangulation has a vertex for every triangle. There are two types of edges. The finite edges join two adjacent triangles, and have direction orthogonal to the common edge of the triangles. The infinite edges start at the triangles adjacent to the boundary of the large triangle $\mathrm{conv}((d,0),(0,d),(0,0))$, and have direction orthogonal to the external edge. This is defined up to the lengths of the finite edges.

Example: In the example above, a dual graph for the regular triangulation is shown in Figure 2.

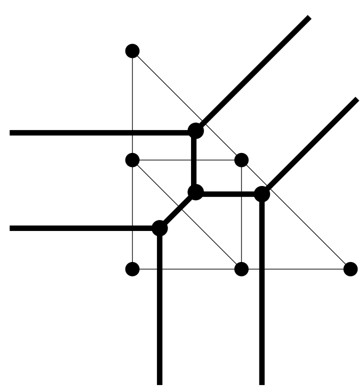

FIGURE 2.

(d) Draw a dual graph to the regular triangulation of $\mathcal{A}_2$ corresponding to $f = tx^2 + xy + t^3y^2 + x + ty + 1$.
(e) Draw a dual graph to the regular triangulation of $\mathcal{A}_1$ corresponding to $f = t^5x + t^3y + t^10$.
(f) Draw a dual graph to the regular triangulation of $\mathcal{A}_3$ corresponding to $f = t^3x^3 + tx^2y + txy^2 + t^3y^3 + tx^2 + xy + ty^2 + tx + ty + t^3$.
(g) Let $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^ay^b$ with $c_{d0}, c_{0d}, c_{00} \neq 0$. Show that the tropical curve defined by f is the image under $x \mapsto -x$ of a dual graph to the regular triangulation defined by f.
(h) Check the previous claim for the examples of the first question.

(i) Conclude that for sufficiently general f there are d tentacles pointing in each direction. What can you say about the genericity condition? What happens in the other cases?

Varieties

(1) If you haven't already done so, read a proof of the Nullstellensatz. Suggested references: Cox, Little, O'Shea, or Eisenbud's commutative algebra course.
(2) Show that if $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_r \rangle$ then $V(f_1, \dots, f_s) = V(g_1, \dots, g_r)$.
(3) Show that $V(I) \cap V(J) = V(I + J)$.
(4) Show that $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.
(5) Show that $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$.
(6) Let $I = \langle x^2, xy^3, y^2z, z^4 \rangle \subset \Bbbk[x, y, z]$. Compute $\sqrt{I}$. What are the irreducible components of $V(I)$?
(7) Show that the Zariski topology is a topology.
(8) Describe the subvariety of $\mathbb{P}^3$ defined by the ideal $I = \langle x_0x_2 - x_1x_3, x_0x_2 - x_1^2, x_1x_3 - x_2^2 \rangle$. Repeat for $I = \langle x_2^2 - x_1x_3, x_1^2 - x_0x_2, x_1x_2x_3 - x_0x_3^2, x_0x_1x_2 - x_0^2x_3 \rangle$. Explain what you notice. (You may find a computer algebra system helps here - ask around until you find a fellow student who knows how to use one if you don't).
(9) Show that if X is an affine or projective variety or a subvariety of a torus, then there is a largest ideal $I \subset S$ with $X = V(I)$ in the sense that if $X = V(J)$ then $J \subseteq I$.
(10) What is the dimension of the affine variety $V(I)$ for $I = \langle x_1, x_2 \rangle \subset \mathbb{A}^5$? What about the affine variety $V(x_1^2 - 3x_2x_3) \subset \mathbb{A}^3$? What is the dimension of the projective variety $V(\langle x_0x_2 - x_1x_3, x_0x_2 - x_1^2, x_1x_3 - x_2^2 \rangle)$?
(11) Let $I = \langle x_1^2 + x_2, x_2^2 + x_3 \rangle \subset \Bbbk[x_1, x_2, x_3]$. What is the multiplicity of the intersection of the affine varieties $V(I)$ and $V(x_i)$ for $i = 1, 2, 3$?
(12) Show that there is a unique curve of degree two through any five points in $\mathbb{P}^2$ in general position. What is the genericity condition?