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# ANALYSIS OF A DIFFUSE INTERFACE APPROACH TO AN ADVECTION DIFFUSION EQUATION ON A MOVING SURFACE

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A diffuse interface model for an advection diffusion equation on a moving surface is formulated involving a small parameter $\varepsilon$ related to the thickness of the interfacial layer. The coefficient functions degenerate on the boundary of the diffuse interface. In appropriately weighted Sobolev spaces, existence and uniqueness of weak solutions is shown. Using energy methods the convergence of solutions to the diffuse interface model to the solution to the equation on the moving surface as $\varepsilon \to 0$ is proved. The approach is intended to be applied to phase field models describing the surface motion. Among other problems we have surfactants on liquid-liquid interfaces and species diffusion on moving grain boundaries in mind.

*Keywords*: Moving surface; surface partial differential equation; weighted Sobolev space; asymptotic analysis.

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## 1. Introduction

Surface quantities subject to partial differential equations on moving hypersurfaces may arise in many applications ranging from fluid dynamics (surfactants on fluid-fluid interfaces[1,18]) over biological systems (lipids on biomembranes[19]) to materials science (species diffusion along grain boundaries[13,12,20]). In this paper we consider prescribed motion of a hypersurface and present and analyze a diffuse interface model to approximate a linear advection diffusion equation.

Let $\{\Gamma(t)\}_{t\in(0,T)}$ denote a moving oriented hypersurface in $\mathbb{R}^d$ that is moving with normal velocity $V(t)\boldsymbol{\nu}(t) : \Gamma(t) \to \mathbb{R}^d$ where $\boldsymbol{\nu}(t)$ is the unit normal to $\Gamma(t)$. Clearly, for describing the purely geometric motion of $\Gamma(t)$ it is sufficient to prescribe the normal velocity, but we also want to take advection along the surface into account and therefore allow for tangential contributions to the velocity field, $\mathbf{v}_{\boldsymbol{\tau}}$. We denote by $\mathbf{v} := V\boldsymbol{\nu} + \mathbf{v}_{\boldsymbol{\tau}}$ the velocity of material points on the surface. Let $c(t) : \Gamma(t) \to \mathbb{R}$ be a scalar conserved quantity for which we postulate that on each (material) portion $G \subset \Gamma$ moving with velocity $\mathbf{v}$ and with unit co-normal $\boldsymbol{\mu}$ on $\partial G$

$$\frac{d}{dt}\left(\int_G c d\mathcal{H}^{d-1}\right)\bigg|_t = -\int_{\partial G(t)} \mathbf{q}(t) \cdot \boldsymbol{\mu}(t) d\mathcal{H}^{d-1}, \tag{1.1}$$

where $\mathbf{q}$ is a tangential dissipative flux. Source and reaction terms are neglected. We assume that $\mathbf{q}$ is minus the surface gradient of $c$. This yields the following strong surface pde[9]:

$$\partial_t^\bullet c + c\nabla_\Gamma \cdot \mathbf{v} - \Delta_\Gamma c = 0. \tag{1.2}$$

Here, $\nabla_\Gamma$ is the tangential surface gradient accounting for variations along $\Gamma(t)$, $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ is the surface Laplace operator, and $\partial_t^\bullet = \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative. The latter is the derivative when following the trajectories given by $\mathbf{v}$ which lie on $\Gamma$. The above surface pde is supplied with initial values $c(t = 0) = \bar{c}$. In this study we will consider closed hypersurfaces.

Our aim is to approximate the above equation (1.2) in the form of a bulk equation holding in a layer around $\Gamma$ of a thickness (almost) proportional to a small length scale $\varepsilon$ (we allow for small deviations). Let $\{\Gamma_\varepsilon(t)\}_{t\in I}$ denote such a layer to which the velocity field, now denoted by $\mathbf{v}_\varepsilon$, is extended in a suitable way. In this thin domain we consider the equation

$$\partial_t(\rho_\varepsilon c_\varepsilon) + \mathbf{v}_\varepsilon \cdot \nabla(\rho_\varepsilon c_\varepsilon) + \rho_\varepsilon c_\varepsilon \nabla \cdot \mathbf{v}_\varepsilon - \nabla \cdot (\rho_\varepsilon \nabla c_\varepsilon) = 0. \tag{1.3}$$

This means that $\rho_\varepsilon c_\varepsilon$ is a bulk conserved quantity involving a dissipative flux of the form $-\rho_\varepsilon \nabla c_\varepsilon$. The function $\rho_\varepsilon$ is a weight that is positive within the layer but vanishes on its spatial boundary $\{\partial\Gamma_\varepsilon(t)\}_t$.

To try such a narrow band approach is motivated by modeling and numerics. It can be used in more complicated applications where the surface is unknown and phase-field methods are applied to model the surface motion as, e.g., in Refs. 6 and 5. In such models, a phase-field variable, $\phi$, changes its value across a thin layer from one prescribed value to another, and this layer defines a diffuse surface. Our approach gives an answer on how to set up an equation, using a suitable function $\rho = \rho(\phi)$, for a surface quantity in such a situation. Also apart from the phase field methodology the approach may turn out useful since in many applications moving hypersurfaces are the limiting case of moving structures which are indeed thin in one direction.

We remark that such a function $\rho$ appeared naturally in a phase field model of diffusion induced grain boundary motion[13,8] and was applied specifically for approximation purposes in Refs. 22 and 23. As in Refs. 13 and 8 and in contrast to Refs. 22 and 23, we choose $\rho_\epsilon$ to have compact support in the layer $\Gamma_\varepsilon$. This has computational advantages in that the equation for $c_\varepsilon$ is solved in a narrow band. Such approximations arise naturally when the diffuse interface motion is given by the double obstacle phase field model proposed in Refs. 4 and 5 for which the diffuse interface is of finite thickness. Another interesting narrow band approximation method[25] is to choose $\rho_\epsilon$ to be the characteristic function of a layer with thickness order $\epsilon$.

A different approach involving bulk equations is to solve the surface partial differential equation on all level sets of a prescribed function. This is inherently an Eulerian method and yields degenerate equations. See Refs. 3, 15, 14, 10 for stationary surfaces and Refs. 1 and 27 for evolving surfaces. On the other hand, an Eulerian approach to transport and diffusion on evolving surfaces was given in Ref. 11. A narrow band numerical formulation for surface elliptic equations was presented in Ref. 7.

We also note that direct discretizations of (1.2) require evolving meshes following the interface as described, e.g., in Refs. 9 and 12. In contrast, the bulk equation may be solved on a fixed bulk mesh, more precisely, at a given time, on those mesh points within the thin interfacial layer. An advantage of the diffuse interface methods is that topological changes of the surface are naturally captured. Apart from the question of whether continuum mechanical models are valid around such events, numerical sharp interface methods typically necessitate severe modification and adaptivity of the data structures which is avoided in the diffuse interface approach.

Previous work on the $\epsilon$-limit of semi-linear parabolic equations on thin domains has considered the continuity of dynamics on fixed flat[16,17] and curved[21] domains. Our analysis comprises the weak solvability of the degenerate equation (1.3) on an evolving thin domain and then the sharp interface analysis as $\varepsilon \to 0$. We consider a moving closed curve embedded in $\mathbb{R}^2$ that is smoothly parametrized at all times over the interval $(0, 2\pi)$ with periodic boundary conditions. The extension to arbitrary space dimensions $d$ is possible but only requires some more technical work.[21] An obvious restriction is that splitting and coalescence events of the moving curve involving topological changes cannot be handled in this analysis.

Precise assumptions and problem statements are given in Sec. 2. In Sec. 3, existence and uniqueness of a weak solution to (1.3) and continuous dependence on the initial values is proved. To deal with the weight $\rho_\varepsilon$ we work on weighted Sobolev spaces as investigated in Ref. 2. Uniform bounds of the $c_\varepsilon$ are derived so that they converge to a function $c$ which is shown to fulfill (1.2). This asymptotic analysis, contained in Sec. 4, follows the lines of Ref. 24 but allows to consider moving surfaces and degenerating weights $\rho_\varepsilon$. Moreover, the formal analysis in Ref. 22 is now rigorously justified in an even more general context. In a concluding Sec. 5 we make some motivating remarks on the assumptions.

## 2. Definitions and Precise Problem Statements

### 2.1. *Assumptions and notation*

#### 2.1.1. *Evolution of the surface*

Let $I = [0, T)$ with some $T > 0$ be a time interval. We consider smooth closed curves $\Gamma(t)$ embedded into $\mathbb{R}^2$ that smoothly depend on time. Let $\Gamma = \{\{t\} \times \Gamma(t)\}_t$. As remarked in the introduction we want to consider advection along the curve for which a smooth velocity field $\mathbf{v} : \Gamma \to \mathbb{R}^2$ is given such that the trajectories lie on $\Gamma$.

The evolving curve is parametrized by a smooth function $\boldsymbol{\gamma} : I \times (0, 2\pi) \to \Gamma$ periodic with respect to the second variable such that $g(t,s) := |\partial_s \boldsymbol{\gamma}(t,s)| \geq 2\lambda > 0$ for all $(t,s) \in \overline{I} \times \overline{(0,2\pi)}$ with a constant $\lambda > 0$. Let $\boldsymbol{\tau} = \partial_s \boldsymbol{\gamma} / |\partial_s \boldsymbol{\gamma}| =: (\tau_1, \tau_2)$ denote the associated unit tangential vector and $\boldsymbol{\nu} := (\tau_2, -\tau_1) = \boldsymbol{\tau}^\perp$ the unit normal. The normal velocity of the curve given in terms of $\boldsymbol{\gamma}$ must be consistent with the velocity field, i.e.

$$\partial_t \boldsymbol{\gamma} \cdot \boldsymbol{\nu} = V = \mathbf{v} \cdot \boldsymbol{\nu}. \tag{2.1}$$

2.1.2. *Diffuse interface*

We further suppose that a family of functions $\rho_\varepsilon \in C^2(I \times \mathbb{R}^2)$ is given that depends continuously on a parameter $\varepsilon \in (0, \bar{\varepsilon})$ with some $\bar{\varepsilon} > 0$. The diffuse interface regions approximating the curves are defined by $\Gamma_\varepsilon := \{\{t\} \times \Gamma_\varepsilon(t)\}_{t \in I}$ where $\Gamma_\varepsilon(t) := \{\rho_\varepsilon(t) > 0\}$. The notion of approximation is that the functions $\rho_\varepsilon$ are such that, as $\varepsilon \to 0$, the sets $\Gamma_\varepsilon(t)$ converge to the curves $\Gamma(t)$ with respect to the Hausdorff distance uniformly in time and linearly in $\varepsilon$.

Let $\Theta := (0, 2\pi) \times (-1, 1)$. The parametrization of the curve leads to a parametrization of $\Gamma_\varepsilon$ in the following way:

$$\Gamma_\varepsilon(t) = \{\boldsymbol{\gamma}_\varepsilon(t,s,z) | (s,z) \in \Theta\}, \quad \boldsymbol{\gamma}_\varepsilon(t,s,z) := \boldsymbol{\gamma}(t,s) + \varepsilon z q(t,s,z,\varepsilon) \boldsymbol{\nu}(t,s).$$

Here, $q$ is a smooth function such that

$$q - 1 \to 0 \quad \text{in } C^3(I \times \Theta) \quad \text{as } \varepsilon \to 0. \tag{2.2}$$

Hence, the parametrization $\boldsymbol{\gamma}_\varepsilon$ is also smooth.

We denote by $dl = |\partial_s \boldsymbol{\gamma}(t,s)| ds$ the length element of the curve $\Gamma(t)$. The scalar curvature $\kappa(t,s)$ is defined by the formula $\partial_l \boldsymbol{\tau} = \kappa \boldsymbol{\nu}$ or $\partial_l \boldsymbol{\nu} = -\kappa \boldsymbol{\tau}$. As a consequence, $\partial_s \boldsymbol{\nu} = -|\partial_s \boldsymbol{\gamma}| \kappa \boldsymbol{\tau}$. Let us state some formulas for the derivatives of $\boldsymbol{\gamma}_\varepsilon$,

$$\begin{aligned} \partial_s \boldsymbol{\gamma}_\varepsilon &= |\partial_s \boldsymbol{\gamma}|(1 - \varepsilon z q \kappa) \boldsymbol{\tau} + \varepsilon z \partial_s q \boldsymbol{\nu}, \\ \partial_z \boldsymbol{\gamma}_\varepsilon &= \varepsilon (q + z \partial_z q) \boldsymbol{\nu}, \\ \partial_{tz} \boldsymbol{\gamma}_\varepsilon &= \varepsilon (\partial_t (q + z \partial_z q) \boldsymbol{\nu} + (q + z \partial_z q) \partial_t \boldsymbol{\nu}). \end{aligned} \tag{2.3}$$

Moreover,

$$\det(\nabla_{(s,z)} \boldsymbol{\gamma}_\varepsilon) = \varepsilon g_\varepsilon \quad \text{with } g_\varepsilon = |\partial_s \boldsymbol{\gamma}|(1 - \varepsilon z q \kappa)(q + z \partial_z q) \tag{2.4}$$

and we assume that $\bar{\varepsilon}$ is small enough such that $g_\varepsilon \geq \lambda$.

For a function $f : \Gamma_\varepsilon \to \mathbb{R}$ on the physical space we can now define its counterpart $\tilde{f}$ on the parameter space via $\tilde{f}(t,s,z) := f(t, \boldsymbol{\gamma}_\varepsilon(t,s,z))$. Observe that

$$\partial_t \tilde{f}(t,s,z) = \frac{d}{dt} f(t, \boldsymbol{\gamma}(t,s,z)) = \partial_t f(t, \boldsymbol{\gamma}(t,s,z)) + \partial_t \boldsymbol{\gamma} \cdot \nabla f(t, \boldsymbol{\gamma}(t,s,z)).$$

To transform spatial derivatives we need the derivatives of the coordinates $(s,z) \in \Theta$ considered as functions of $\mathbf{x} \in \Gamma_\varepsilon(t)$. By the inverse function theorem

$$\nabla \begin{pmatrix} s \\ z \end{pmatrix} = (\nabla_{(s,z)} \boldsymbol{\gamma}_\varepsilon)^{-1} = \frac{1}{\varepsilon g_\varepsilon} \begin{pmatrix} \varepsilon(q + z\partial_z q)\boldsymbol{\tau}^\perp \\ g(1-\varepsilon z q\kappa)\boldsymbol{\nu}^\perp - \varepsilon z \partial_s q \boldsymbol{\tau}^\perp \end{pmatrix}.$$

Hence, $\nabla f = \partial_s \tilde{f} \nabla s + \partial_z \tilde{f} \nabla z$ where

$$\nabla s = \frac{1}{g(1-\varepsilon z q\kappa)} \boldsymbol{\tau}, \quad \nabla z = \frac{1}{\varepsilon} \frac{1}{q + z\partial_z q} \boldsymbol{\nu} - \frac{z\partial_s q}{g_\varepsilon} \boldsymbol{\tau}.$$

Furthermore, if $f$ is a function on the moving curve $\Gamma$, then

$$\nabla_\Gamma f = \boldsymbol{\tau} \frac{\partial_s \tilde{f}}{|\partial_s \boldsymbol{\gamma}|} = \boldsymbol{\tau} \frac{\partial_s \tilde{f}}{g}.$$

In the following, with a slight abuse of notation the tilde on functions like $f$ will be dropped for convenience.

Next, we assume that there is a function $\bar{\rho} : (-1,1) \to \mathbb{R}$ and there are constants $C_2 \geq C_1 > 0$ such that

$$\sup_{(t,s,z)} |\rho_\varepsilon(t,s,z) - \bar{\rho}(z)| \to 0 \quad \text{and} \quad \sup_{(t,s,z)} |\partial_t \rho_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.5}$$

$$C_2 \leq \frac{\rho_\varepsilon(t,s,z)}{\bar{\rho}(z)} \leq C_1 \quad \text{and} \quad |\partial_t \rho_\varepsilon(t,s,z)|, |\partial_{tt} \rho_\varepsilon(t,s,z)| \leq C_2 \bar{\rho}(z) \quad \forall\, t,s,z,\varepsilon. \tag{2.6}$$

The function $\bar{\rho}$ is a non-negative differentiable weight function bounded by a positive constant with $\bar{\rho}(z) > 0$ if $z \in (-1,1)$ but which vanishes if $|z| = 1$. We also assume that it is normalized in the sense that

$$\int_{-1}^{1} \bar{\rho}(z) dz = 1. \tag{2.7}$$

We assume that there is a smooth extension of $\mathbf{v}$ to a field $\mathbf{v}_\varepsilon : \Gamma \to \mathbb{R}^2$ such that for a constant $C > 0$

$$|\mathbf{v}_\varepsilon(t,s,z) - \mathbf{v}(t,s)| \leq C\varepsilon, \quad |\partial_t \mathbf{v}_\varepsilon(t,s,z) - \partial_t \mathbf{v}(t,s)| \leq C\varepsilon \quad \forall\, t,s,z,\varepsilon. \tag{2.8}$$

Observe that then thanks to the consistency assumption (2.1)

$$\boldsymbol{\nu} \cdot (\mathbf{v}_\varepsilon - \partial_t \boldsymbol{\gamma}_\varepsilon) = \boldsymbol{\nu} \cdot (\mathbf{v}_\varepsilon - \mathbf{v}) + \boldsymbol{\nu} \cdot (\mathbf{v} - \partial_t \boldsymbol{\gamma} - \varepsilon z \partial_t (q \boldsymbol{\nu})) = O(\varepsilon). \tag{2.9}$$

For the initial data we assume that $\bar{c} \in H^1_{\mathrm{per}}((0,2\pi))$.

### 2.2. *Weighted Sobolev spaces*

Since $\bar{\rho}(\pm 1) = 0$, the coefficients in (1.3) degenerate towards the boundary of the interfacial layer. To overcome this problem, weighted Sobolev spaces can be used. Consider the Borel measure

$$\omega_{\bar{\rho}}(A) := \int_A \bar{\rho}(s,z) dz ds$$

on Lebesgue-measurable sets $A \subset \Theta$. The space

$$L^2(\Theta, \omega_{\bar\rho}) := \{f : \Theta \to \mathbb{R} \,|\, f\, \omega_{\bar\rho}\text{-measurable}, \|f\|_{L^2(\Theta,\omega_{\bar\rho})} < \infty\},$$

where

$$\|f\|_{L^2(\Theta,\omega_{\bar\rho})} := \left(\int_\Theta \bar\rho |f|^2 dzds\right)^{1/2}$$

is complete and a Hilbert space with the scalar product

$$(f,g)_{L^2(\Theta,\omega_{\bar\rho})} := \int_\Theta fg d\omega_{\bar\rho} = \int_\Theta \bar\rho fg dzds.$$

Since $1/\bar\rho \in L^1_{\mathrm{loc}}(\Theta)$ we have that $L^2(\Theta,\omega_{\bar\rho}) \subset L^1_{\mathrm{loc}}(\Theta)$ (see Prop. 2.1 in Ref. 2 or the references therein). For a function $f \in L^2(\Theta,\omega_{\bar\rho})$ we can therefore define a derivative in a distributional sense. The $\bar\rho$-weighted Sobolev space $H^1(\Theta,\omega_{\bar\rho})$ is defined to be the set of all functions $f \in L^2(\Theta,\omega_{\bar\rho})$ such that the distributional derivatives $\partial_s f, \partial_z f$ belong to $L^2(\Theta,\omega_{\bar\rho})$ again, i.e. are weak derivatives. It is a Hilbert space with the scalar product

$$(f,g)_{H^1(\Theta,\omega_{\bar\rho})} := \int_\Theta \bar\rho(fg + \nabla_{(s,z)} f \cdot \nabla_{(s,z)} g) dzds.$$

On every set $A \subset\subset \Theta$ the function $\bar\rho$ is bounded from below by a positive constant. Hence, $L^2(A,\omega_{\bar\rho})$ coincides with the usual Lebesgue space $L^2(A)$. Thanks to this, one can show that the smooth functions are dense in $L^2(\Theta,\omega_{\bar\rho})$ and $H^1(\Theta,\omega_{\bar\rho})$. A similar argument is also used in the following lemma[2] which is repeated here for convenience.

**Lemma 2.1.** *The embedding $H^1(\Theta,\omega_{\bar\rho}) \hookrightarrow L^2(\Theta,\omega_{\bar\rho})$ is compact.*

**Proof.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\Theta,\omega_{\bar\rho})$, w.l.o.g. bounded by 1, and let $\zeta > 0$ be an arbitrary small real number.

For $\delta \in (0,1)$ define $\Theta^\delta := \{(s,z) \in \Theta | z \in (-1+\delta, 1-\delta)\}$ and

$$f_n^\delta(s,z) := \begin{cases} f_n(s,z) & \text{if } (s,z) \in \Theta^\delta, \\ 0 & \text{else.} \end{cases}$$

Since the $f_n$ are bounded in $L^2(\Theta,\omega_{\bar\rho})$ we have for the error of this cutoff that

$$\begin{aligned} \int_\Theta \bar\rho |f_n - f_n^\delta|^2 dzds &= \int_{\Theta\setminus\Theta^\delta} \bar\rho |f_n|^2 \\ &\le \sup_{|z|\ge 1-\delta} \bar\rho(z) \mathrm{Vol}(\Theta\setminus\Theta^\delta) \|f_n\|^2_{L^2(\Theta,\omega_{\bar\rho})} \\ &< \frac{\zeta^2}{4} \end{aligned} \tag{2.10}$$

for all $n$ if $\delta$ is small enough, which is assumed for the following.

On $\Theta^\delta$ the function $\bar\rho$ is bounded from below by a constant $\bar\rho_0^\delta > 0$, hence

$$\|f_n^\delta\|^2_{H^1(\Theta)} \le \frac{1}{\bar\rho_0^\delta} \|f_n^\delta\|^2_{H^1(\Theta,\omega_{\bar\rho})} \le \frac{1}{\bar\rho_0^\delta}.$$

Let $R \in \mathbb{R}$ denote an upper bound of $\bar{\rho}$. Since the embedding $H^1(\Theta) \to L^2(\Theta)$ is compact there is a number $N_\zeta \in \mathbb{N}$ and there are $N_\zeta$ functions $g_i \in L^2(\Theta) \subset L^2(\Theta, \omega_{\bar{\rho}}), i = 1, \ldots, N_\zeta$, such that for all $n \in \mathbb{N}$ there is an index $i$ with $\|f_n^\delta - g_i\|_{L^2(\Theta)} < \frac{1}{\sqrt{R}}\frac{\zeta}{2}$. Therefore

$$\|f_n^\delta - g_i\|_{L^2(\Theta,\omega_{\bar{\rho}})} \leq \left(\int_\Theta R|f_n^\delta - g_i|^2 dzds\right)^{1/2} < \frac{\zeta}{2}.$$

Together with (2.10) this means that for every $n \in \mathbb{N}$ there is an index $i \in \{1, \ldots, N_\zeta\}$ such that $\|f_n - g_i\|_{L^2(\Theta,\omega_{\bar{\rho}})} < \zeta$. □

We introduce the spaces

$$X := \{f \in H^1(\Theta, \omega_{\bar{\rho}}) | f \text{ periodic in } s\},$$
$$B := \{f \in L^2(\Theta, \omega_{\bar{\rho}}) | f \text{ periodic in } s\}$$

and will consider the spaces $L^2(I; X)$ and $L^2(I; B)$ with the generic norms

$$\|f\|_{L^2(I;X)} := \left(\int_0^T \|f(t)\|_X^2\right)^{1/2}, \quad \|f\|_{L^2(I;B)} := \left(\int_0^T \|f(t)\|_B^2\right)^{1/2}.$$

### 2.3. *Problem formulations*

#### 2.3.1. *Equation on the evolving curve*

We multiply (1.2) by a test function $\chi$ and integrate, first over $\Gamma(t)$ and then with respect to time. After that, we partially integrate with respect to space (recall that the curves are closed) and transform to the space $I \times (0, 2\pi)$:

$$\begin{aligned}
0 &= \int_0^T \int_{\Gamma(t)} (\partial_t^\bullet c + c\nabla_\Gamma \cdot \mathbf{v} - \Delta_\Gamma c)\chi d\mathcal{H}^1 dt \\
&= \int_0^T \int_{\Gamma(t)} \Bigg(\partial_t c\chi + \nabla c \cdot \partial_t \boldsymbol{\gamma}\chi + \underbrace{\nabla c \cdot (\mathbf{v} - \partial_t \boldsymbol{\gamma})}_{=\nabla_\Gamma c\cdot(\mathbf{v}-\partial_t\boldsymbol{\gamma}) \text{ by } (2.1)} \chi \\
&\quad + c\chi\nabla_\Gamma \cdot (\mathbf{v} - \partial_t\boldsymbol{\gamma}) + c\chi\nabla_\Gamma \cdot \partial_t\boldsymbol{\gamma} + \nabla_\Gamma c \cdot \nabla_\Gamma \chi\Bigg) d\mathcal{H}^1 dt \\
&= \int_0^T \int_{\Gamma(t)} ((\partial_t c + \partial_t\boldsymbol{\gamma} \cdot \nabla c)\chi - c\nabla_\Gamma\chi \cdot (\mathbf{v} - \partial_t\boldsymbol{\gamma}) \\
&\quad + c\chi\nabla_\Gamma \cdot \partial_t\boldsymbol{\gamma} + \nabla_\Gamma c \cdot \nabla_\Gamma\chi) d\mathcal{H}^1 dt \\
&= \int_0^T \int_0^{2\pi} \left(\partial_t\tilde{c}\tilde{\chi}g + \partial_s\tilde{c}\tilde{\chi}\boldsymbol{\tau} \cdot (\tilde{\mathbf{v}} - \partial_t\boldsymbol{\gamma}) + \tilde{c}\tilde{\chi}\boldsymbol{\tau} \cdot \partial_s\tilde{\mathbf{v}} + \partial_s\tilde{c}\partial_s\tilde{\chi}\frac{1}{g}\right) dsdt.
\end{aligned}$$

We perform a partial integration with respect to time in the first term and arrive at

**Problem 2.1.** *Find $c \in L^2(I; H^1_{\mathrm{per}}((0,2\pi)))$ such that*

$$\begin{aligned} 0 = & \int_0^{2\pi} \bar{c}\chi(0)g(0)ds - \int_0^T \int_0^{2\pi} c\partial_t(\chi g)dsdt \\ & + \int_0^T \int_0^{2\pi} \left( -c\partial_s\chi\boldsymbol{\tau}\cdot(\mathbf{v} - \partial_t\boldsymbol{\gamma}) + c\chi\boldsymbol{\tau}\cdot\partial_{st}\boldsymbol{\gamma} + \partial_s c\partial_s\chi\frac{1}{g}\right)dsdt \end{aligned} \tag{2.11}$$

*for all $\chi \in L^2(I; H^1_{\mathrm{per}}((0,2\pi)))$ with $\partial_t\chi \in L^2(I; L^2_{\mathrm{per}}((0,2\pi)))$ and $\chi(T) = 0$.*

2.3.2. *Diffuse interface approximation*

The procedure is similar in the diffuse interface setting. Boundary terms do not occur during the partial integration since $\rho_\varepsilon$ vanishes there. We obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\Gamma_\varepsilon(t)} (\partial_t^\bullet(\rho_\varepsilon c_\varepsilon) + \rho_\varepsilon c_\varepsilon\nabla\cdot\mathbf{v}_\varepsilon - \nabla\cdot(\rho_\varepsilon\nabla c_\varepsilon))\chi d\mathbf{x}dt \\ &= \int_0^T \int_{\Gamma_\varepsilon(t)} (\partial_t(\rho_\varepsilon c_\varepsilon)\chi + \nabla(\rho_\varepsilon c_\varepsilon)\cdot\partial_t\boldsymbol{\gamma}_\varepsilon\chi + \nabla(\rho_\varepsilon c_\varepsilon)\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon)\chi \\ &\quad + \rho_\varepsilon c_\varepsilon\chi\nabla\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon) + \rho_\varepsilon c_\varepsilon\chi\nabla\cdot\partial_t\boldsymbol{\gamma}_\varepsilon + \rho_\varepsilon\nabla c_\varepsilon\cdot\nabla\chi)d\mathbf{x}dt \\ &= \int_0^T \int_{\Gamma_\varepsilon(t)} ((\partial_t(\rho_\varepsilon c_\varepsilon) + \partial_t\boldsymbol{\gamma}_\varepsilon\cdot\nabla(\rho_\varepsilon c_\varepsilon))\chi - \rho_\varepsilon c_\varepsilon\nabla\chi\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon) \\ &\quad + \rho_\varepsilon c_\varepsilon\chi\nabla\cdot\partial_t\boldsymbol{\gamma}_\varepsilon + \rho_\varepsilon\nabla c_\varepsilon\cdot\nabla\chi)d\mathbf{x}dt \\ &= \int_0^T \int_\Theta (\partial_t(\rho_\varepsilon c_\varepsilon)\chi g_\varepsilon + \rho_\varepsilon c_\varepsilon(\partial_s\chi\nabla s + \partial_z\chi\nabla z)\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon)g_\varepsilon \\ &\quad + \rho_\varepsilon c_\varepsilon\chi(\nabla s\cdot\partial_s\partial_t\boldsymbol{\gamma}_\varepsilon + \nabla z\cdot\partial_z\partial_t\boldsymbol{\gamma}_\varepsilon)g_\varepsilon \\ &\quad + \rho_\varepsilon(\partial_s c_\varepsilon\nabla s + \partial_z c_\varepsilon\nabla z)\cdot(\partial_s\chi\nabla s + \partial_z\chi\nabla z)g_\varepsilon)\varepsilon dzdsdt. \end{aligned}$$

Using the formulas for $\nabla s$ and $\nabla z$, multiplying with $1/\varepsilon$, partially integrating with respect to time in the first term and defining the coefficient functions

$$\begin{aligned} a_0 &:= \frac{\rho_\varepsilon}{\bar{\rho}}g_\varepsilon, \\ a_1 &:= \frac{\sqrt{\rho_\varepsilon g_\varepsilon}}{\sqrt{\bar{\rho}}g(1-\varepsilon zq\kappa)}, \\ a_2 &:= \frac{\rho_\varepsilon g_\varepsilon}{\bar{\rho}(q + z\partial_z q)}, \\ b_0 &:= \frac{\partial_t\rho_\varepsilon g_\varepsilon}{\bar{\rho}} + \frac{(q+z\partial_z q)\rho_\varepsilon}{\bar{\rho}}\boldsymbol{\tau}\cdot\partial_{st}\boldsymbol{\gamma}_\varepsilon + \left(\frac{g(1-\varepsilon zq\kappa)\rho_\varepsilon}{\bar{\rho}}\frac{1}{\varepsilon}\boldsymbol{\nu} - z\partial_s q\boldsymbol{\tau}\right)\cdot\partial_{zt}\boldsymbol{\gamma}_\varepsilon, \\ b_1 &:= -\frac{(q+z\partial_z q)\rho_\varepsilon}{\bar{\rho}}\boldsymbol{\tau}\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon), \end{aligned}$$

$$b_2 := \left(-\frac{g(1-\varepsilon zq\kappa)\rho_\varepsilon}{\bar{\rho}}\frac{1}{\varepsilon}\boldsymbol{\nu} + \frac{z\partial_s q\rho_\varepsilon}{\bar{\rho}}\boldsymbol{\tau}\right)\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon),$$

$$b_3 := \frac{z\partial_s q\sqrt{\rho_\varepsilon}}{\sqrt{\bar{\rho}}g_\varepsilon}.$$

We finally obtain

**Problem 2.2.** *Find $c_\varepsilon \in L^2(I;X)$ such that*

$$\begin{aligned} 0 = &\int_\Theta \bar{\rho}\bar{c}\chi(0)a_0(0)dzds - \int_0^T\int_\Theta \bar{\rho}c_\varepsilon\partial_t(\chi a_0)dzdsdt \\ &+ \int_0^T\int_\Theta \bar{\rho}\Big(b_0c_\varepsilon\chi + b_1c_\varepsilon\partial_s\chi + b_2c_\varepsilon\partial_z\chi \\ &+ (a_1\partial_s c_\varepsilon - b_3\partial_z c_\varepsilon)(a_1\partial_s\chi - b_3\partial_z\chi) + \frac{1}{\varepsilon^2}a_2\partial_z c_\varepsilon\partial_z\chi\Big)dzdsdt \end{aligned} \tag{2.12}$$

*for all $\chi \in L^2(I;X)$ with $\partial_t\chi \in L^2(I;B)$ and $\chi(T) = 0$.*

## 3. Analysis of the $\varepsilon$ Problem

The linear Problem 2.2 can be solved by proceeding as in the case without weight. In fact, the essential detail is the compactness of the embedding $X \hookrightarrow B$ which has been provided in Lemma 2.1. Before presenting an existence and uniqueness result let us first briefly discuss the coefficient functions in (2.12).

By the smoothness of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_\varepsilon$, the quantities $\boldsymbol{\tau}, \boldsymbol{\nu}, g_\varepsilon$ and $\kappa$ are also smooth. By (2.3), (2.8) and its consequence (2.9) the terms $\frac{1}{\varepsilon}\boldsymbol{\nu}\cdot\partial_{zt}\boldsymbol{\gamma}_\varepsilon$ and $\frac{1}{\varepsilon}\boldsymbol{\nu}\cdot(\mathbf{v}_\varepsilon - \partial_t\boldsymbol{\gamma}_\varepsilon)$ as well as their time derivatives are of order $O(\varepsilon^0)$. Hence, thanks to the assumptions (2.6), (2.5) and (2.8) all the coefficient functions $a_i, b_j$ and their time derivatives are uniformly bounded and continuous. The assumption (2.6), the positivity of $g_\varepsilon$ as assumed below (2.4), and (2.2) furthermore imply that the coefficients $a_i$ are uniformly bounded from below by positive constants. We stress that all these constants are independent of $\varepsilon$, which will turn out to be useful in the next section.

**Theorem 3.1.** *Under the assumptions stated in Sec. 2 and if $\varepsilon$ is small enough there is a unique solution $c_\varepsilon \in L^2(I;X)\cap H^1(I;B)$ to Problem 2.2 which satisfies the estimates*

$$\sup_{t\in I}\int_\Theta \bar{\rho}|c_\varepsilon(t)|^2dzds + \|\partial_s c_\varepsilon\|^2_{L^2(I;B)} + \frac{1}{\varepsilon^2}\|\partial_z c_\varepsilon\|^2_{L^2(I;B)} \le C\int_0^{2\pi}\bar{c}^2ds, \tag{3.1}$$

$$\sup_{t\in I}\int_\Theta \bar{\rho}\left(|\partial_s c_\varepsilon(t)|^2 + \frac{1}{\varepsilon^2}|\partial_z c_\varepsilon(t)|^2\right)dzds + \|\partial_t c_\varepsilon\|^2_{L^2(I;B)} \le C, \tag{3.2}$$

*with a positive constant $C$ independent of $\varepsilon$.*

**Proof.** In the following, the $C_i, i = 1, 2, \ldots$, denote positive constants independent of $(t, s, z, \varepsilon)$. One may argue with a time discretization. For a number $N \in \mathbb{N}$ let $\tau = T/2^N$ and $t_n^N := n\tau, n = 0, \ldots, 2^N$. Set $c_0^N := \bar{c}, a_{0,n}^N := a_0(t_n^N)$ and similarly for the other coefficient functions. Now, consider the subsequent problems for $n = 1, \ldots, 2^N$: find $c_n^N \in X$ such that

$$0 = \int_\Theta \bar{\rho}\bigg(a_{0,n}^N \frac{c_n^N - c_{n-1}^N}{\tau}\chi + b_{0,n}^N c_n^N \chi + b_{1,n}^N c_n^N \partial_s \chi + b_{2,n}^N c_n^N \partial_z \chi + (a_{1,n}^N \partial_s c_n^N - b_{3,n}^N \partial_z c_n^N)(a_{1,n}^N \partial_s \chi - b_{3,n}^N \partial_z \chi) + \frac{1}{\varepsilon^2} a_{2,n}^N \partial_z c_n^N \partial_z \chi\bigg) dzds \tag{3.3}$$

for all $\chi \in X$. The Lax–Milgram theorem can be applied to show that (3.3) has a unique solution. To obtain a coercive operator it may be necessary to reduce $\tau$ and $\varepsilon$, but the properties of the coefficient functions allow one to find appropriate values independently of $n$ and $N$.

We may insert $\chi = c_n^N$ in (3.3), multiply with $\tau$ and sum up for $n = 1, \ldots, \bar{n}$ with some $\bar{n} \leq 2^N$. Observe that the first term gives

$$\begin{aligned}
&\sum_{n=1}^{\bar{n}} \tau \int_\Theta \bar{\rho} a_{0,n}^N \frac{c_n^N - c_{n-1}^N}{\tau} c_n^N \\
&\quad \geq \sum_{n=1}^{\bar{n}} \bigg(\frac{1}{2}\int_\Theta \bar{\rho} a_{0,n}^N (c_n^N)^2 - \frac{1}{2}\int_\Theta \bar{\rho} a_{0,n}^N (c_{n-1}^N)^2\bigg) \\
&\quad = \frac{1}{2}\int_\Theta \bar{\rho} a_{0,\bar{n}}^N (c_{\bar{n}}^N)^2 - \frac{1}{2}\int_\Theta \bar{\rho} a_{0,0}^N (c_0^N)^2 + \sum_{n=1}^{\bar{n}} \tau \int_\Theta \bar{\rho} \frac{a_{0,n-1}^N - a_{0,n}^N}{2\tau}(c_{n-1}^N)^2 \\
&\quad \geq C_1 \|c_{\bar{n}}^N\|_B^2 - C_2 \|\bar{c}\|_B^2 - C_3 \sum_{n=0}^{\bar{n}-1} \tau \|c_n^N\|_B^2
\end{aligned}$$

thanks to the properties of $a_0$, in particular its positivity. Together with the other terms in (3.3) one can derive

$$\|c_{\bar{n}}^N\|_B^2 + \sum_{n=1}^{\bar{n}} \tau \bigg(\|\partial_s c_n^N\|_B^2 + \frac{1}{\varepsilon^2}\|\partial_z c_n^N\|_B^2\bigg) \leq C_4 \|\bar{c}\|_B^2 + C_5 \sum_{n=0}^{\bar{n}-1} \tau \|c_n^N\|_B^2.$$

A Gronwall argument yields

$$\sup_{n \in \{1,\ldots,2^N\}} \|c_n^N\|_B^2 + \sum_{n=1}^{2^N} \tau \bigg(\|\partial_s c_n^N\|_B^2 + \frac{1}{\varepsilon^2}\|\partial_z c_n^N\|_B^2\bigg) \leq C_6 \|\bar{c}\|_B^2. \tag{3.4}$$

In order to obtain an estimate for time shifts we may furthermore test (3.3) with $(c_n^N - c_{n-1}^N)/\tau$. Clearly for the first term

$$\sum_{n=1}^{\bar{n}} \tau \int_\Theta \bar{\rho} a_{0,n}^N \bigg|\frac{c_n^N - c_{n-1}^N}{\tau}\bigg|^2 \geq C_7 \sum_{n=1}^{\bar{n}} \tau \bigg\|\frac{c_n^N - c_{n-1}^N}{\tau}\bigg\|_B^2. \tag{3.5}$$

Next we observe that

$$\begin{aligned}
&\sum_{n=1}^{\bar{n}}\tau\int_\Theta \bar{\rho} a_{2,n}^N \partial_z c_n^N \partial_z \frac{c_n^N - c_{n-1}^N}{\tau} \\
&\quad \geq \sum_{n=1}^{\bar{n}}\frac{1}{2}\int_\Theta \bar{\rho}(a_{2,n}^N|\partial_z c_n^N|^2 - a_{2,n}^N|\partial_z c_{n-1}^N|^2) \\
&\quad = \int_\Theta \bar{\rho} a_{2,\bar{n}}^N|\partial_z c_{\bar{n}}^N|^2 - \int_\Theta \bar{\rho} a_{2,0}^N|\partial_z c_0^N|^2 + \sum_{n=1}^{\bar{n}}\tau\int_\Theta \bar{\rho}\,\frac{a_{2,n-1}^N - a_{2,n}^N}{2\tau}\,|\partial_z c_{n-1}^N|^2 \\
&\quad \geq C_8\|\partial_z c_{\bar{n}}^N\|_B^2 - C_9\|\underbrace{\partial_z \bar{c}}_{=0}\|_B^2 - C_{10}\sum_{n=1}^{\bar{n}}\tau\|\partial_z c_n^N\|_B^2. \qquad (3.6)
\end{aligned}$$

The last term can be estimated by (3.4). Furthermore, we have that

$$\begin{aligned}
&\sum_{n=1}^{\bar{n}}\tau\int_\Theta \bar{\rho} b_{2,n}^N c_n^N \partial_z \frac{c_n^N - c_{n-1}^N}{\tau} \\
&\quad = \sum_{n=1}^{\bar{n}}\int_\Theta \bar{\rho}(b_{2,n}^N c_n^N \partial_z c_n^N - b_{2,n-1}^N c_{n-1}^N \partial_z c_{n-1}^N + (b_{2,n-1}^N c_{n-1}^N - b_{2,n}^N c_n^N)\partial_z c_{n-1}^N) \\
&\quad = \int_\Theta \bar{\rho}\left(b_{2,\bar{n}}^N c_{\bar{n}}^N \partial_z c_{\bar{n}}^N - b_{2,0}^N c_0^N \underbrace{\partial_z c_0^N}_{=\partial_z\bar{c}=0}\right) \\
&\qquad + \sum_{n=1}^{\bar{n}}\tau\int_\Theta \bar{\rho}\left(b_{2,n}^N \frac{c_{n-1}^N - c_n^N}{\tau} + \frac{b_{2,n-1}^N - b_{2,n}^N}{\tau}c_{n-1}^N\right)\partial_z c_{n-1}^N \\
&\quad \geq -\delta\int_\Theta \bar{\rho}|\partial_z c_{\bar{n}}^N|^2 - C_{11}\int_\Theta \bar{\rho}|c_{\bar{n}}^N|^2 \\
&\qquad - \delta\sum_{n=1}^{\bar{n}}\tau\int_\Theta \bar{\rho}\left|\frac{c_n^N - c_{n-1}^N}{\tau}\right|^2 - C_{12}\sum_{n=1}^{\bar{n}}\tau\int_\Theta \bar{\rho}(|\partial_z c_{n-1}^N|^2 + |c_n^N|^2),
\end{aligned}$$

where $\delta > 0$ is so small such that $C_7 - \delta > 0$ and $C_8 - \delta > 0$ (eventually even smaller, taking further terms into account). The remaining terms can be handled similarly and finally we see that

$$\begin{aligned}
&\sum_{n=1}^{\bar{n}}\tau\left\|\frac{c_n^N - c_{n-1}^N}{\tau}\right\|_B^2 + \|\partial_s c_{\bar{n}}^N\|_B^2 + \frac{1}{\varepsilon^2}\|\partial_z c_{\bar{n}}^N\|_B^2 \\
&\quad \leq C_{13} + C_{14}\sum_{n=1}^{\bar{n}}\tau(\|c_n^N\|_B^2 + \|\partial_z c_n^N\|_B^2 + \|\partial_s c_n^N\|_B^2).
\end{aligned}$$

In view of (3.4) we infer that

$$\sum_{n=1}^{2^N} \tau \left\| \frac{c_n^N - c_{n-1}^N}{\tau} \right\|_B^2 + \sup_{n\in\{1,\dots,2^N\}} \left( \|\partial_s c_n^N\|_B^2 + \frac{1}{\varepsilon^2}\|\partial_z c_n^N\|_B^2 \right) \leq C_{15}. \tag{3.7}$$

Define the functions $c^N, \hat{c}^N \in L^2(I;X)$ by

$$c^N(t) := \frac{t - t_{n-1}^N}{\tau} c_n^N + \frac{t_n^N - t}{\tau} c_{n-1}^N, \quad c^{N+}(t) := c_n^N, \quad \text{if } t \in (t_{n-1}^N, t_n^N].$$

We will now use test functions $\chi^N \in X^N \subset L^2(I;X)$ of the form $\chi^N(t) = \chi_n^N \in X$ for all $t \in (t_{n-1}^N, t_n^N]$. Observe that $X^M \subset X^N$ for $M \leq N$. Analogously, the functions $a_i^N$ and $b_j^N$ are defined. It follows directly from (3.3) that

$$\begin{aligned} 0 = \sum_{n=1}^{2^N} \int_\Theta \bar{\rho} \Big( & a_0^N \partial_t c^N \chi^M + b_0^N c^{N+} \chi^M + b_1^N c^{N+} \partial_s \chi^M + b_2^N c^{N+} \partial_z \chi^M \\ & + (a_1^N \partial_s c^{N+} - b_3^N \partial_z c^{N+})(a_1^N \partial_s \chi^M - b_3^N \partial_z \chi^M) + \frac{1}{\varepsilon^2} a_2^N \partial_z c^{N+} \partial_z \chi^M \Big) dz ds \end{aligned} \tag{3.8}$$

for all $\chi^M \in X^M$ with $M \leq N$.

By the estimates (3.4) and (3.7) there is a function $c \in L^\infty(I;X) \cap H^1(I;B) \times (\hookrightarrow C^0(I;B)$ compact[26]) such that

$$\begin{aligned} c^N, c^{N+} &\overset{*}{\rightharpoonup} c && \text{in } L^\infty(I;X), \\ c^N &\to c && \text{in } C^0(I;B), \\ \partial_t c^N &\rightharpoonup \partial_t c && \text{in } L^2(I;B), \end{aligned}$$

for a subsequence as $N \to \infty$. From (3.4) and (3.7) we also see that the estimates (3.1) and (3.2) are fulfilled. The approximations of the coefficient functions converge in $C^0(I \times \Theta)$. Going to the limit in (3.8) therefore yields (2.12) for all $\chi^M \in X^M$, $M \in \mathbb{N}$. With a density argument and after partial integration with respect to time we see that $c$ indeed fulfills (2.12).

The uniqueness follows directly from estimate (3.1). □

## 4. Asymptotic Analysis

For the following convergence theorem, the so-called energy methods are applied.

**Theorem 4.1.** *As $\varepsilon \to 0$, the solutions $c_\varepsilon$ to Problem 2.2 converge in $C^0(I;B)$ to a function $c$ with the following properties*:

(1) $\partial_z c = 0$, *hence $c = c(t,s)$ can be considered as a function on $I \times (0, 2\pi)$*,
(2) $c \in L^2(I; H^1_{\text{per}}((0,2\pi)))$ *solves Problem 2.1.*

**Proof.** By Ref. 26, Cor. 4, the embedding $L^2(I;X) \cap H^1(I;B) \hookrightarrow C^0(I;B)$ is compact. The key estimates (3.1) and (3.2) imply that there is a function $c \in L^2(I;X) \cap H^1(I;B)$ such that

$$\begin{aligned} c_\varepsilon &\rightharpoonup c && \text{in } L^2(I;X), \\ \partial_t c_\varepsilon &\rightharpoonup \partial_t c && \text{in } L^2(I;B), \\ c_\varepsilon &\to c && \text{in } C^0(I;B) \quad \text{and almost everywhere} \end{aligned}$$

for a subsequence as $\varepsilon \to 0$. Since by (3.1) $\frac{1}{\varepsilon}\partial_z c_\varepsilon$ is bounded in $L^2(I;B)$ we additionally have that $\partial_z c_\varepsilon \to 0$ in $L^2(I;B)$, whence $\partial_z c = 0$. This means that $c = c(t,s) \in L^2(I; H^1_{\text{per}}((0,2\pi))) \cap H^1(I; L^2_{\text{per}}((0,2\pi)))$.

Concerning the coefficients in (2.12) we immediately deduce the following convergence statements as $\varepsilon \to 0$ : $a_0 \to g, a_1 \to 1/\sqrt{g}, a_2 \to g, b_1 \to -\boldsymbol{\tau} \cdot (\mathbf{v} - \partial_t \boldsymbol{\gamma})$, and $b_3 \to 0$ in $C^0([0,T]; C^0(\Theta))$. The first term in $b_0$ converges to zero thanks to (2.5), which also implies that $\partial_t a_0 \to \partial_t g$. For the last one observe that by (2.3) $\frac{1}{\varepsilon}\boldsymbol{\nu} \cdot \partial_{tz}\boldsymbol{\gamma}_\varepsilon = \partial_t q + z\partial_{tz} q \to 0$ so that altogether $b_0 \to \boldsymbol{\tau} \cdot \partial_{st}\boldsymbol{\gamma}$.

Consider now test functions $\chi \in L^2(I;X) \cap H^1(I;B)$ with $\partial_z \chi = 0$ and $\chi(T) = 0$ in (2.12). The above convergence statements yield

$$\begin{aligned} 0 &= \int_\Theta \bar\rho \bar c \chi(0) a_0(0) dz ds - \int_0^T \int_\Theta \bar\rho (c_\varepsilon \partial_t \chi a_0 + c_\varepsilon \chi \partial_t a_0) dz ds dt \\ &\quad + \int_0^T \int_\Theta \bar\rho (b_0 c_\varepsilon \chi + b_1 c_\varepsilon \partial_s \chi + (a_1 \partial_s c_\varepsilon - b_3 \partial_z c_\varepsilon) a_1 \partial_s \chi) dz ds dt \\ &\to \int_\Theta \bar\rho \bar c \chi(0) g(0) dz ds - \int_0^T \int_\Theta \bar\rho c \partial_t (\chi g) dz ds dt \\ &\quad + \int_0^T \int_\Theta \bar\rho \left( \boldsymbol{\tau} \cdot \partial_{st}\boldsymbol{\gamma} c \chi - \boldsymbol{\tau} \cdot (\mathbf{v} - \partial_t \boldsymbol{\gamma}) c \partial_s \chi + \frac{1}{g} \partial_s c \partial_s \chi \right) dz ds dt. \end{aligned}$$

Apart from $\bar\rho$ all terms in the last two lines do not depend on $z$ any more. By

$$\int_\Theta \bar\rho(z) \bar c(s) \chi(0,s) g(0,s) dz ds = \underbrace{\int_{-1}^1 \bar\rho(z) dz}_{=1 \text{ by } (2.7)} \int_0^{2\pi} \bar c(s) \chi(0,s) g(0,s) ds$$

and proceeding analogously with the other terms we see that $c$ indeed solves Problem 2.1. In Ref. 9 it is shown that there is a unique weak solution to Problem 2.1. As a consequence, the whole set of function $\{c_\varepsilon\}_\varepsilon$ converges to $c$ as stated above. □

## 5. Discussion and Remarks

We have shown the existence and uniqueness of a weak solution to (1.3) by transforming the moving domain $\Gamma_\varepsilon$ to a fixed (in time) parameter space and using a suitably weighted Sobolev space to deal with the function $\rho_\varepsilon$. Further we have proved that these solutions $c_\varepsilon$ converge to a weak solution to (1.2) as $\varepsilon \to 0$. The estimate on $\frac{1}{\varepsilon}\partial_z c_\varepsilon$ is essential to obtain a limiting function fulfilling $\partial_z c = 0$ which means that variations in the direction normal to the hypersurface vanish in the limit. We conclude with several remarks.

### 5.1. *Possible extensions of the results*

In the case of open curves one has to prescribe boundary conditions for $c$ on $\partial\Gamma$ to close (1.2). The parametrization must then reflect the fact that the boundary points move with velocity $\mathbf{v}$, hence $\partial_t \boldsymbol{\gamma}(t,s) = \mathbf{v}(t,s)$ for $s \in \{0, 2\pi\}$. An extension to hypersurfaces of higher dimension is possible, too. Parametrizing $\Gamma$ over a reference manifold $\mathcal{M}$ the derivatives with respect to $s$ become weighted surface gradients $\nabla_{\mathcal{M}}$, cf. Ref. 21. In all these cases the set up in normal direction and the form of $\gamma_\varepsilon$ are not affected.

### 5.2. *Choice of the profile*

In the phase field approach with double-obstacle potentials in order to describe the moving surface,[4,5] to leading order the phase field variable $\phi$ has a sinusoidal profile in the normal direction to the interface. For $\rho$, of particular interest is a profile of the form $1 - \phi^2$,

$$\bar{\rho}(z) = \frac{2}{\pi}(1 - \sin(z))(1 + \sin(z)).$$

This function grows like the squared distance to $\pm 1$ close to the boundary $(0, 2\pi) \times \{\pm 1\} \subset \partial\Theta$. Our hope is that the degeneracy of $\bar{\rho}$ turns out to be helpful in numerical simulations. It keeps the mass of the surface quantity in the diffuse interfacial region independently of the extension of the velocity field away from the sharp interface. To see this, we integrate (1.3) over $\Gamma_\varepsilon(t)$ for general $\rho_\epsilon$ and apply a transport identity. Recall that the motion field for $t \mapsto \partial\Gamma_\varepsilon(t)$ is $\partial_t \boldsymbol{\gamma}_\varepsilon$ rather than $\mathbf{v}_\varepsilon$.

$$\begin{aligned}
0 &= \int_{\Gamma_\varepsilon(t)} (\partial_t(\rho_\varepsilon c_\varepsilon) + \mathbf{v}_\varepsilon \cdot \nabla(\rho_\varepsilon c_\varepsilon) + \rho_\varepsilon c_\varepsilon \nabla \cdot \mathbf{v}_\varepsilon - \nabla \cdot (\rho_\varepsilon \nabla c_\varepsilon)) d\mathcal{H}^{d-1} \\
&= \int_{\Gamma_\varepsilon(t)} (\partial_t(\rho_\varepsilon c_\varepsilon) + \partial_t \boldsymbol{\gamma}_\varepsilon \cdot \nabla(\rho_\varepsilon c_\varepsilon) + \rho_\varepsilon c_\varepsilon \nabla \cdot \partial_t \boldsymbol{\gamma}_\varepsilon \\
&\quad + \nabla \cdot (\rho_\varepsilon c_\varepsilon (\mathbf{v}_\varepsilon - \partial_t \boldsymbol{\gamma}_\varepsilon) - \rho_\varepsilon \nabla c_\varepsilon)) d\mathcal{H}^{d-1} \\
&= \frac{d}{dt}\left(\int_{\Gamma_\varepsilon} \rho_\varepsilon c_\varepsilon d\mathcal{H}^{d-1}\right)\bigg|_t + \int_{\partial\Gamma_\varepsilon(t)} \rho_\varepsilon (c_\varepsilon(\mathbf{v}_\varepsilon - \partial_t \boldsymbol{\gamma}_\varepsilon) - \nabla c_\varepsilon) \cdot \boldsymbol{\nu}_{\partial\Gamma_\varepsilon(t)} d\mathcal{H}^{d-2}.
\end{aligned}$$

Since $\frac{1}{\varepsilon}\int_{\Gamma_\varepsilon} \rho_\varepsilon c_\varepsilon \to \int_\Gamma \bar{c}$ it is desirable that $\frac{d}{dt}(\int_{\Gamma_\varepsilon} \rho_\varepsilon c_\varepsilon) = 0$. Choosing a uniformly positive $\bar{\rho}$ one needs other requirements in order for the flux over the boundary to vanish. In more complex applications the diffuse interfacial domain $\Gamma_\varepsilon$ as well as the velocity field $\mathbf{v}_\varepsilon$ may be unknown and subject to other pdes so that, in general, $\mathbf{v}_\varepsilon - \partial_t \boldsymbol{\gamma}_\varepsilon \neq 0$ on $\partial\Gamma_\varepsilon(t)$. Consequently, there is a Robin boundary condition for $c_\varepsilon$ which may be difficult to implement in simulations. The degenerating $\rho_\varepsilon$ elegantly circumvents this condition.

### 5.3. *Initial conditions*

We simply extended $\bar{c}$ constantly in $z$, which is natural in view of the fact that the diffusivity in $z$ direction is fast, scaling with $1/\varepsilon^2$. Choosing another extension results in the function $\underline{c} : s \mapsto \int_{-1}^{1} \bar{\rho}(z)\bar{c}(s,z)dz$ replacing $\bar{c}$ in the first term of (2.11) from the asymptotic analysis. A requirement to approximate the originating problem is then clearly that $\underline{c} = \bar{c}$.

### 5.4. *Source terms and reactions*

In the identity (1.1) to derive the equation for $c$ on the moving surface, source terms of the form $\int_G f$ on the right-hand side with a given function $f$ defined on $\Gamma$ can easily be taken into account and lead to the additional term $-f$ on the left-hand side of (1.2). In the corresponding equation (1.3) on the diffuse interface the additional term reads $-\rho_\varepsilon f_\varepsilon$ where $f_\varepsilon$ is a suitable extension of $f$ away from $\Gamma$ defined similarly to $\mathbf{v}_\varepsilon$, for example extended constantly in normal direction. Under appropriate regularity assumptions on $f$ and its extension both the analysis and the asymptotic analysis can still be established analogously as presented. Reaction terms are left for future research.

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