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FULLY DISCRETE FINITE ELEMENT APPROXIMATION FOR ANISOTROPIC SURFACE DIFFUSION OF GRAPHS*

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Abstract. We analyze a fully discrete numerical scheme for approximating the evolution of graphs for surfaces evolving by anisotropic surface diffusion. The scheme is based on the idea of second order operator splitting for the nonlinear geometric fourth order equation. This yields two coupled spatially second order problems, which are approximated by linear finite elements. The time discretization is semi-implicit. We prove error bounds for the resulting scheme and present numerical test calculations that confirm our analysis and illustrate surface diffusion.

Key words. surface diffusion, anisotropic, geometric motion, second order operator splitting, nonlinear partial differential equation, finite element, fully discrete, error estimates, fourth order parabolic equation

AMS subject classifications. 65N30, 35K55

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1. Introduction.

This article is concerned with the geometric problem of determining an evolving surface $\Gamma(t)$ whose motion is governed by the highly nonlinear fourth order geometric anisotropic surface diffusion equation

$$V = \Delta_\Gamma \mathcal{H}_\gamma \quad \text{on } \Gamma(t), \tag{1.1}$$

where V and Δ_Γ denote, respectively, the normal velocity and the Laplace–Beltrami (surface Laplacian) operator for $\Gamma(t)$. Furthermore, $\mathcal{H}_\gamma$ denotes the anisotropic mean curvature of the surface with respect to the positive, convex, and 1-homogeneous surface energy density $\gamma : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}$. We can introduce $\mathcal{H}_\gamma$ formally as the first variation of the surface energy

$$A_\gamma(\Gamma) = \int_\Gamma \gamma(\nu), \tag{1.2}$$

where ν denotes the unit normal to Γ.

Modelling morphological surface evolution and growth is fundamental in materials science and the study of microstructure. The surface evolution law (1.1) is referred to as surface diffusion because it models the diffusion of mass within the bounding surface of a solid body. At the atomistic level atoms on the surface move along the

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surface due to a driving force consisting of a chemical potential difference. For a surface with surface energy density $\gamma(\nu)$ the appropriate chemical potential in this setting is the anisotropic curvature $\mathcal{H}_\gamma$. This leads to the flux law

$$\rho V = -\operatorname{div}_\Gamma \mathbf{j},$$

where ρ is the mass density and $\mathbf{j}$ is the mass flux in the surface, with the constitutive flux law [19], [21]

$$\mathbf{j} = -D\nabla_\Gamma \mathcal{H}_\gamma.$$

Here, D is the diffusion constant. From these equations we obtain the law (1.1) after an appropriate nondimensionalization. The notion of surface diffusion is due to Mullins [21] and for a review we refer the reader to [5].

Our sign convention is that $\mathcal{H}_\gamma$ with respect to the outer normal is positive for the Wulff shape $\mathcal{W} := \{p \in \mathbb{R}^{n+1} \mid \langle p, q\rangle \le \gamma(q)\ \forall q \in \mathbb{R}^{n+1}\}$.

This evolution has interesting geometrical properties: if $\Gamma(t)$ is a closed surface bounding a domain $\Omega(t)$, then the volume of $\Omega(t)$ is preserved and the surface energy (or weighted surface area) of $\Gamma(t)$ decreases. The corresponding result in the graph case is given in Lemma 2.2. At present, the existence and uniqueness theory for surface diffusion is limited to the isotropic case $\gamma(q) := |q|$, $q \in \mathbb{R}^{n+1}$. For example, it is known that for closed curves in the plane or closed surfaces in $\mathbb{R}^3$ balls are asymptotically stable subject to small perturbations; see [15], [17]. However, topological changes such as pinch-off are possible [18], [20], and a one-dimensional graph may lose its graph property in finite time whilst the surface evolves smoothly [16].

In what follows we shall study evolving surfaces $\Gamma(t)$ which can be described, for each $t \ge 0$, as the graph of a height function $u(\cdot, t)$ over some base domain $\Omega \subset \mathbb{R}^n$, i.e., $\Gamma(t) = \{(x, u(x,t)) \in \mathbb{R}^{n+1} \mid x \in \Omega\}$. The area element and a unit normal, denoted by $Q(u)$ and $\nu(u)$, are then given by

$$Q(u) = \sqrt{1 + |\nabla u|^2}, \qquad \nu(u) = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}} = \frac{(\nabla u, -1)}{Q(u)}$$

so that we can calculate the surface energy or weighted area for a graph Γ given by the height function u as

$$A_\gamma(\Gamma) = \mathcal{I}_\gamma(u) := \int_\Omega \gamma(\nu(u))Q(u) = \int_\Omega \gamma(\nabla u, -1)$$

in view of the homogeneity of γ. Thus the first variation of A_γ in the direction of a function $\phi \in C_0^\infty(\Omega)$ is

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{I}_\gamma(u+\epsilon\phi)_{|\epsilon=0} &= \sum_{i=1}^n \int_\Omega \gamma_{p_i}(\nabla u, -1)\phi_{x_i} = -\sum_{i,j=1}^n \int_\Omega \gamma_{p_i p_j}(\nabla u, -1) u_{x_i x_j}\phi \\ &= -\int_\Omega \mathcal{H}_\gamma \phi = \int_\Omega w\phi,\end{aligned}$$

where we use $-w$ to denote the anisotropic or weighted mean curvature of the surface in the graph case so that

$$w := -\sum_{i,j=1}^n \gamma_{p_i p_j}(\nabla u, -1) u_{x_i x_j}. \tag{1.3}$$

In order to translate (1.1) into a differential equation for $u = u(x,t)$, we observe that the normal velocity V of $\Gamma(t)$ is given by $V = -\frac{u_t}{Q(u)}$. Furthermore, if $v : \Omega \to \mathbb{R}$, then the Laplace–Beltrami operator on $\Gamma(t)$ is given by (see (2.5) below)

$$\Delta_\Gamma v = \frac{1}{Q(u)} \nabla \cdot \left(\left(Q(u) I - \frac{\nabla u \otimes \nabla u}{Q(u)} \right) \nabla v \right),$$

where $\otimes$ denotes the usual tensor product of two vectors in $\mathbb{R}^n$. Thus, anisotropic surface diffusion for graphs is defined by the following highly nonlinear fourth order evolutionary equation:

$$u_t = -\nabla \cdot \left(\left(Q(u) I - \frac{\nabla u \otimes \nabla u}{Q(u)} \right) \nabla \left(\sum_{i,j=1}^{n} \gamma_{p_i p_j}(\nabla u, -1) u_{x_i x_j} \right) \right). \tag{1.4}$$

The aim of this paper is to analyze a fully discrete finite element approximation of the initial-boundary value problem in the case of *graphs*. We use the second order splitting method for fourth order problems proposed by Elliott, French, and Milner [14] for the fourth order Cahn–Hilliard equation and subsequently employed for surface diffusion by Deckelnick, Dziuk, and Elliott [12]. Thus the space discretization is accomplished using H^1 conforming finite element spaces. For example, continuous piecewise linear elements on triangulations are sufficient. On the other hand, in time we use a novel semi-implicit discretization which requires only the solution of linear algebraic equations but which preserves the Liapunov structure. This ensures the natural stability properties of the scheme with a time step independent of the spatial mesh size. The scheme involves stabilizing the explicit Euler scheme by adding a semi-implicit linear form which involves the discrete time derivative. This stabilizing form has two terms. One involves the anisotropy and is designed to yield a stable linearization. The second term is of higher order with respect to the time step and is based on the Laplace–Beltrami form. It is designed to yield the L^2 stability bound, (3.11), on the discrete solution similar to that enjoyed by the solution of the partial differential equation. A similar idea was previously used in [11] for the anisotropic mean curvature flow of graphs and in [23] for surface diffusion. The main achievement of the paper is the derivation of a priori geometric error bounds. We prove optimal order bounds for the difference of the normals measured in the L^2 norm over either the continuous surface $\Gamma(t)$ or the discrete surface $\Gamma_h(t)$ and the L^2 norm on the discrete surface of the difference of the tangential gradients of the anisotropic mean curvature. This latter bound is equivalent to an H^{-1} bound on the difference in normal velocities. Some numerical computations are presented which confirm the analysis and which illustrate the effect of anisotropy.

A second order splitting finite element scheme for axially symmetric surfaces was presented by Coleman, Falk, and Moakher [7], [8] together with some stability results and interesting numerical computations illustrating pinch-off and the formation of beads. A first finite element error analysis for the second order splitting method for surface diffusion in the axially symmetric case was presented by Deckelnick, Dziuk, and Elliott [12]. Subsequently, Bänsch, Morin, and Nochetto [1] developed an optimal order continuous in time finite element error analysis for the second order splitting method in the case of multidimensional graphs. Our work has the distinctive feature of analyzing a fully discrete second order splitting finite element method for nonlinear surface anisotropy using a stable semi-implicit time stepping scheme.

Remark 1.1. The analysis is easily extended to the more general evolution law

$$V = \Delta_\Gamma(\mathcal{H}_\gamma - f) + g, \tag{1.5}$$

where f is a force arising from an extra term in the energy and g is a surface growth term. For example, including mechanical energy leads to the appearance of f and in epitaxial growth g models the deposition of atoms.

Remark 1.2. Our results are presented for zero Neumann boundary conditions with exact quadrature. The results and arguments also hold without change for the case of Ω being a box and periodic boundary conditions. Minor modifications are required for homogeneous Dirichlet boundary conditions. These three sets of conditions have the property of being variationally separated and allow the second order splitting method to work.

Remark 1.3. The approach to surface diffusion in this paper is entirely analogous to the work of Elliott, French, and Milner [14] for the Cahn–Hilliard equation where u is an order or phase field variable and w is the chemical potential. The variational gradient flow structure is identical in each setting. Indeed the degenerate Cahn–Hilliard equation yields a diffuse interface approximation to surface diffusion [4].

The paper is organized as follows. In section 2 we introduce some notation and assumptions. We set up the numerical scheme and derive some preliminary estimates in section 3, whilst section 4 contains the proof of the error bounds. Finally, section 5 contains some numerical results.

2. Notation and assumptions.

2.1. Differential geometry. Let Γ be a C^2 hypersurface in $\mathbb{R}^{n+1}$ with unit normal ν. For any function $\bar\eta = \bar\eta(x_1, \dots, x_{n+1})$ defined in a neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of Γ we define its tangential gradient on Γ by

$$\nabla_\Gamma \bar\eta := D\bar\eta - \langle D\bar\eta, \nu\rangle \nu,$$

where on $\mathbb{R}^{n+1}$ $\langle\cdot,\cdot\rangle$ denotes the usual scalar product and $D\bar\eta$ denotes the usual gradient. The tangential gradient $\nabla_\Gamma\bar\eta$ depends only on the values of $\bar\eta$ on Γ and $\langle \nabla_\Gamma\bar\eta, \nu\rangle = 0$. The Laplace–Beltrami operator on Γ is defined as the tangential divergence of the tangential gradient, i.e.,

$$\Delta_\Gamma\bar\eta = \langle \nabla_\Gamma, \nabla_\Gamma\bar\eta\rangle.$$

Let Γ have a boundary $\partial\Gamma$ whose intrinsic unit outer normal, tangential to Γ, is denoted by μ. Then the surface Green's formula is

$$\int_\Gamma \langle \nabla_\Gamma\bar\xi, \nabla_\Gamma\bar\eta\rangle = \int_{\partial\Gamma} \bar\xi \langle \nabla_\Gamma\bar\eta, \mu\rangle - \int_\Gamma \bar\xi \Delta_\Gamma\bar\eta. \tag{2.1}$$

We now turn to the situation in hand where $\Gamma(t) = \{(x, u(x,t)) \in \mathbb{R}^{n+1} \mid x \in \Omega\}$. For functions $v = v(x)$, $x \in \Omega$, we use the extension $\bar v(x, x_{n+1}) = v(x)$ and define

$$\nabla_\Gamma v := \nabla_\Gamma \bar v = D\bar v - \langle D\bar v, \nu(u)\rangle \nu(u) = P(\nu(u)) D\bar v,$$

where we observe that $D\bar v = (\nabla v, 0)$, $\nu(u) = (\nabla u, -1)/Q(u)$ and $P(\nu(u))$ is given by

$$P(\nu(u)) := I - \nu(u) \otimes \nu(u).$$

Here, we have used the tensor product notation $y \otimes y := yy^T$. It follows that

$$\langle \nabla_\Gamma v, \nabla_\Gamma \eta \rangle = \nabla v \cdot \nabla \eta - \frac{1}{Q(u)^2} \nabla v \cdot \nabla u \nabla \eta \cdot \nabla u = \frac{1}{Q(u)} (\nabla v)^t E(\nabla u) \nabla \eta, \tag{2.2}$$

where

$$E(\nabla u) := Q(u) I - \frac{\nabla u \otimes \nabla u}{Q(u)}.$$

For later use we note that

$$\langle P(\nu(u)) D\bar{v}, D\bar{w} \rangle Q(u) = (\nabla v)^t E(\nabla u) \nabla w, \tag{2.3}$$

$$(\nabla v)^t E(\nabla u) \nabla v \geq \frac{|\nabla v|^2}{Q(u)}. \tag{2.4}$$

Integrating (2.2) over Γ we derive

$$\int_\Gamma \langle \nabla_\Gamma \bar{v}, \nabla_\Gamma \bar{\eta} \rangle = \int_\Omega \langle \nabla_\Gamma v, \nabla_\Gamma \eta \rangle Q(u) = \int_\Omega (\nabla v)^t E(\nabla u) \nabla \eta.$$

If we combine this relation with (2.1) we obtain for test functions η, which vanish on $\partial\Omega$

$$\int_\Gamma \bar{\eta}\, \Delta_\Gamma \bar{v} = \int_\Omega \eta \nabla \cdot (E(\nabla u) \nabla v) = \int_\Omega \eta \frac{1}{Q(u)} \nabla \cdot (E(\nabla u) \nabla v) Q(u),$$

so that

$$\Delta_\Gamma v := \Delta_\Gamma \bar{v} = \frac{1}{Q(u)} \nabla \cdot (E(\nabla u) \nabla v). \tag{2.5}$$

2.2. The anisotropy. We suppose that $\gamma : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is smooth with $\gamma(p) > 0$ for $p \in \mathbb{R}^{n+1} \setminus \{0\}$ and that γ is positively homogeneous of degree one, i.e.,

$$\gamma(\lambda p) = |\lambda| \gamma(p) \quad \forall\, \lambda \neq 0,\ p \neq 0. \tag{2.6}$$

Here, $|\cdot|$ denotes the Euclidean norm. It is not difficult to verify that (2.6) implies

$$\langle \gamma'(p), p \rangle = \gamma(p), \qquad \langle \gamma''(p) p, q \rangle = 0, \tag{2.7}$$

$$\gamma_{p_i}(\lambda p) = \frac{\lambda}{|\lambda|} \gamma_{p_i}(p), \qquad \gamma_{p_i p_j}(\lambda p) = \frac{1}{|\lambda|} \gamma_{p_i p_j}(p) \tag{2.8}$$

for all $p \in \mathbb{R}^{n+1} \setminus \{0\}$, $q \in \mathbb{R}^{n+1}$, $\lambda \neq 0$, and $i, j \in \{1, \ldots, n+1\}$. Finally, we assume that there exists $\gamma_0 > 0$ such that

$$\langle D^2 \gamma(p) q, q \rangle \geq \gamma_0 |q|^2 \quad \forall\, p, q \in \mathbb{R}^{n+1},\ |p| = 1,\ \langle p, q \rangle = 0. \tag{2.9}$$

Further information about the geometric properties and physical relevance of anisotropic energy functionals can be found, respectively, in [2] and [24].

2.3. Function spaces. By $(\cdot, \cdot)$ we denote the $L^2(\Omega)$ inner product $(v, \eta) := \int_\Omega v(x) \eta(x) dx$ for $v, \eta \in L^2(\Omega)$ with norm $\|v\| := (v, v)^{\frac{1}{2}}$. Also $H^{m,p}(\Omega)$ denotes the usual Sobolev space with the corresponding norm being given by $\|u\|_{H^{m,p}(\Omega)} = (\sum_{k=0}^m \|D^k u\|^p_{L^p(\Omega)})^{\frac{1}{p}}$ with the usual modification for $p = \infty$. For $p = 2$ we simply write $H^m(\Omega) = H^{m,2}(\Omega)$ with norm $\|\cdot\|_{H^m(\Omega)}$.

2.4. The variational formulation and initial-boundary value problem. Rather than discretizing the fourth order equation (1.4) we use the height u of the graph and the anisotropic curvature of the graph w as variables and consider the two second order equations (1.1), (1.3),

$$u_t = \nabla \cdot (E(\nabla u)\nabla w), \tag{2.10}$$

$$w = -\sum_{i,j=1}^{n} \gamma_{p_i p_j}(\nabla u, -1) u_{x_i x_j}. \tag{2.11}$$

The system is closed using Neumann boundary conditions and an initial condition for u.

$$E(\nabla u)\nabla w \cdot \nu_{\partial\Omega} = 0, \tag{2.12}$$

$$\langle \gamma'(\nu(u)), (\nu_{\partial\Omega}, 0)\rangle = 0, \tag{2.13}$$

$$u(\cdot, 0) = u_0. \tag{2.14}$$

The first equation, (2.12), is the zero mass flux condition whereas the second equation, (2.13), is the natural variational boundary condition which defines w as the variational derivative or chemical potential for the surface energy functional. Note that an initial condition on w is not required.

In order to write down the variational formulation it is convenient to introduce the following forms:

Laplace–Beltrami (LB) form,

$$\mathcal{E}(u; w, \eta) := \int_\Omega (\nabla w)^t E(\nabla u)\nabla \eta dx$$

Anisotropic mean curvature (AMC) form,

$$\mathcal{A}(u, \eta) := \sum_{i=1}^{n} \int_\Omega \gamma_{p_i}(\nu(u))\eta_{x_i} dx.$$

Then it is straightforward to show the following equivalence between the classical form of the initial-boundary value problem and the variational formulation.

LEMMA 2.1. *Let $u \in C^1([0,T]; C^4(\bar\Omega))$, $u(\cdot, 0) = u_0$, and $w \in C^0([0,T]; C^2(\bar\Omega))$. Then (u, w) is a solution of* (2.10)–(2.13) *iff $u(\cdot, 0) = u_0$ and the following variational equations are satisfied:*

$$(\partial_t u, \eta) + \mathcal{E}(u; w, \eta) = 0 \quad \forall\, \eta \in H^1(\Omega), \tag{2.15}$$

$$(w, \eta) - \mathcal{A}(u, \eta) = 0 \quad \forall\, \eta \in H^1(\Omega). \tag{2.16}$$

LEMMA 2.2. *The solution (u, w) satisfies for each $t \in [0, T]$ the surface energy equation*

$$\mathcal{I}_\gamma(u) + \int_0^t \mathcal{E}(u; w, w)\, ds = \mathcal{I}_\gamma(u_0) \tag{2.17}$$

and the conservation laws

$$(u, 1) = (u_0, 1), \qquad (w, 1) = 0. \tag{2.18}$$

Furthermore, for each $t \in [0,T]$ we have the bound

$$\|u(t)\|^2 + \int_0^t \|w\|^2\, ds \le C(\gamma, u_0, T). \tag{2.19}$$

Proof. Taking $\eta = w$ in (2.15) and $\eta = \partial_t u$ in (2.16) and subtracting the resulting equations yields (2.17). Taking $\eta = 1$ in (2.15) and (2.16) yields (2.18).

In order to prove the first part of (2.19), we use $\eta = u$ in (2.15) and apply (2.29) which gives

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\mathcal{E}(u; w, u) \le \mathcal{E}(u; w, w)^{\frac{1}{2}}\mathcal{E}(u; u, u)^{\frac{1}{2}} \le \frac{1}{2}\mathcal{E}(u; w, w) + \frac{1}{2}\int_\Omega Q(u).$$

Integrating this inequality with respect to time we obtain with the help of (2.17),

$$\|u(t)\|^2 \le \|u_0\|^2 + \int_0^t \mathcal{E}(u; w, w) ds + \frac{1}{\inf_{|p|=1}\gamma(p)}\int_0^t I_\gamma(u)\, ds \le C(\gamma, u_0, T).$$

Using $\eta = w$ in (2.16) we deduce

$$\|w\|^2 = \mathcal{A}(u, w) \le \sup_{|p|=1}|\gamma'(p)|\int_\Omega |\nabla w| \le C\left(\int_\Omega \frac{|\nabla w|^2}{Q(u)}\right)^{\frac{1}{2}}\left(\int_\Omega Q(u)\right)^{\frac{1}{2}},$$

so that (2.4) and similar arguments as above yield

$$\int_0^t \|w\|^2 ds \le C\int_0^t \mathcal{E}(u; w, w) ds + C\int_0^t Q(u)\, ds \le C(\gamma, u_0, T). \qquad \square$$

Remark 2.3. The surface energy equation (2.17) can be written as

$$\int_{\Gamma(t)} \gamma(\nu) + \int_0^T \int_{\Gamma(t)} |\nabla_\Gamma \mathcal{H}_\gamma|^2 = \int_{\Gamma(0)} \gamma(\nu). \tag{2.20}$$

The conservation of u is equivalent to the conservation of the volume lying below the graph of the surface. That the integral over Ω of the anisotropic mean curvature is zero is a consequence of the fact that constant vertical variations in the height of the graph do not change the anisotropic surface area.

2.5. Geometric lemmas. The following algebraic relations are elementary.

LEMMA 2.4.

$$|\nabla(u - v)|^2 = (Q(u) - Q(v))^2 + |\nu(u) - \nu(v)|^2 Q(u)Q(v), \tag{2.21}$$

$$\left|\frac{1}{Q(u)} - \frac{1}{Q(v)}\right| \le |\nu(u) - \nu(v)|, \tag{2.22}$$

$$|Q(u) - Q(v)| \le Q(u)Q(v)|\nu(u) - \nu(v)|. \tag{2.23}$$

LEMMA 2.5 (properties of the anisotropy and the AMC form $\mathcal{A}$). *Let $u, v \in H^{1,\infty}(\Omega)$. Then*

$$\mathcal{A}(v, u - v) \ge \mathcal{I}_\gamma(u) - \mathcal{I}_\gamma(v) - \bar{\gamma}\int_\Omega |\nu(u) - \nu(v)|^2 Q(u), \tag{2.24}$$

where

$$\bar\gamma := \frac{1}{\sqrt{5}-1}\max\left\{\sup_{|p|=1}|\gamma'(p)|, \sup_{|p|=1}|\gamma''(p)|\right\}. \tag{2.25}$$

If in addition $|\nabla u| \le K$ *a.e. in* Ω, *then*

$$|\mathcal{A}(u,\eta) - \mathcal{A}(v,\eta)| \le C(\gamma, K)\int_\Omega |\nu(u)-\nu(v)||\nabla\eta|. \tag{2.26}$$

Proof. The first inequality follows from the estimate

$$\sum_{i=1}^n \gamma_{p_i}(\nu(v))(u-v)_{x_i} \ge \gamma(\nu(u))Q(u) - \gamma(\nu(v))Q(v) - \bar\gamma|\nu(u)-\nu(v)|^2 Q(u) \tag{2.27}$$

which is contained in the proof of Theorem 3.1 in [11, p. 430]. Let us next turn to (2.26). Lemma 6.1 in [11] implies that there exists $c_0 = c_0(K) > 0$ such that

$$|s\nu(u) + (1-s)\nu(v)| \ge c_0 \quad \text{a.e. in } \Omega \quad \forall\, s \in [0,1]. \tag{2.28}$$

Note that c_0 is independent of v. As a consequence,

$$\begin{aligned}|\gamma_{p_i}(\nu(u)) - \gamma_{p_i}(\nu(v))| &= \left|\sum_{j=1}^{n+1}\int_0^1 \gamma_{p_ip_j}(s\nu(u)+(1-s)\nu(v))ds(\nu_j(u)-\nu_j(v))\right| \\ &\le \frac{1}{c_0}\max_{|p|=1}|D^2\gamma(p)||\nu(u)-\nu(v)| \le C(\gamma,K)|\nu(u)-\nu(v)|,\end{aligned}$$

since $D^2\gamma$ is positively homogeneous of degree -1. This yields (2.26). ∎

LEMMA 2.6 (properties of the LB form $\mathcal{E}$). *Let* $u, v \in H^{1,\infty}(\Omega)$. *Then*

$$|\mathcal{E}(u;w,\eta)| \le \mathcal{E}(u;w,w)^{\frac12}\mathcal{E}(u;\eta,\eta)^{\frac12}. \tag{2.29}$$

If in addition $|\nabla u| \le K$ *a.e. in* Ω, *then*

$$\mathcal{E}(v;u-v,u-v) \le C(K)\int_\Omega |\nu(u)-\nu(v)|^2 Q(v), \tag{2.30}$$

$$|\mathcal{E}(u;\eta_1,\eta_2) - \mathcal{E}(v;\eta_1,\eta_2)| \le C(K)\|\nabla\eta_1\|_\infty \int_\Omega |\nu(u)-\nu(v)||\nabla\eta_2|Q(v), \tag{2.31}$$

$$\begin{aligned}|\mathcal{E}(u;\eta_1,\eta_2) - \mathcal{E}(v;\eta_1,\eta_2)| &\le \epsilon\mathcal{E}(v;\eta_1,\eta_1) \\ &\quad + \frac{C(K)}{\epsilon}\|\nabla\eta_2\|_\infty^2 \int_\Omega |\nu(u)-\nu(v)|^2 Q(v).\end{aligned} \tag{2.32}$$

Proof. Using (2.3) together with Young's inequality we have

$$\begin{aligned}|\mathcal{E}(u;w,\eta)| &= \left|\int_\Omega \langle P(\nu(u))D\bar w, D\bar\eta\rangle Q(u)\right| \\ &\le \int_\Omega \langle P(\nu(u))D\bar w, D\bar w\rangle^{\frac12}\langle P(\nu(u))D\bar\eta, D\bar\eta\rangle^{\frac12} Q(u) \\ &\le \mathcal{E}(u;w,w)^{\frac12}\mathcal{E}(u;\eta,\eta)^{\frac12}.\end{aligned}$$

Next, observing that $(\nabla(u-v),0) = Q(u)\nu(u) - Q(v)\nu(v)$ we obtain

$$\begin{aligned}
&\langle P(\nu(v))(\nabla(u-v),0),(\nabla(u-v),0)\rangle \\
&\quad = \langle (I-(\nu(v)\otimes\nu(v)))(Q(u)\nu(u)-Q(v)\nu(v)),(Q(u)\nu(u)-Q(v)\nu(v))\rangle \\
&\quad = Q(u)^2(1-\langle\nu(u),\nu(v)\rangle^2) = Q(u)^2(1-\langle\nu(u),\nu(v)\rangle)(1+\langle\nu(u),\nu(v)\rangle) \\
&\quad \le Q(u)^2|\nu(u)-\nu(v)|^2,
\end{aligned}$$

since $1-\langle\nu(u),\nu(v)\rangle = \frac{1}{2}|\nu(u)-\nu(v)|^2$. Multiplication of the above inequality by $Q(v)$ followed by integration over Ω yields (2.30). From the definition of $P(\nu(u))$ and (2.23) we infer

$$|P(\nu(u))Q(u) - P(\nu(v))Q(v)| \le C(K)|\nu(u)-\nu(v)|Q(v),$$

which implies (2.31). Finally, writing $D\bar\eta = (\nabla\eta,0)$ and using (2.23) as well as (2.4) we have

$$\begin{aligned}
|\mathcal{E}(u;\eta_1,\eta_2) - \mathcal{E}(v;\eta_1,\eta_2)| &\le \int_\Omega |\langle P(\nu(v))D\bar\eta_1, D\bar\eta_2\rangle||Q(v)-Q(u)| \\
&\quad + \int_\Omega |\langle (P(\nu(v))-P(\nu(u)))D\bar\eta_1, D\bar\eta_2\rangle|Q(u) \\
&\le \int_\Omega \langle P(\nu(v))D\bar\eta_1, D\bar\eta_1\rangle^{\frac{1}{2}}\langle P(\nu(v))D\bar\eta_2, D\bar\eta_2\rangle^{\frac{1}{2}}|\nu(u)-\nu(v)|Q(u)Q(v) \\
&\quad + C(K)\int_\Omega |\nu(u)-\nu(v)|\sqrt{Q(v)}\frac{|\nabla\eta_1|}{\sqrt{Q(v)}}|\nabla\eta_2| \\
&\le \epsilon\mathcal{E}(v;\eta_1,\eta_1) + \frac{C(K)}{\epsilon}\|\nabla\eta_2\|_\infty^2\int_\Omega |\nu(u)-\nu(v)|^2 Q(v).
\end{aligned}$$

This concludes the proof of (2.32). ◻

Remark 2.7. We note that inequalities (2.30) and (2.32) were proved in [1] as Lemmas 4.7 and 4.5, respectively. The argument used above, employing the projection P, is more direct and slightly simpler than the one used in [1] in that it avoids the splitting of Ω into subsets.

LEMMA 2.8. *Let $u,v \in H^{1,\infty}(\Omega)$ with $|\nabla u| \le K$ a.e. in Ω. There exists a constant $c_1 > 0$ which depends only on K and γ_0 from* (2.9) *such that for*

$$D := \int_\Omega (\gamma(\nu(v)) - \langle\gamma'(\nu(u)),\nu(v)\rangle)Q(v)$$

we have

$$D \ge c_1\int_\Omega |\nu(u)-\nu(v)|^2 Q(v).$$

Proof. This is just a reformulation of Lemma 3.2 in [9]. ◻

3. Discretization.

3.1. The finite element approximation. We now turn to the discretization of (2.15), (2.16). Let $\mathcal{T}_h$ be a family of triangulations of Ω with maximum mesh size $h := \max_{\tau\in\mathcal{T}_h}\mathrm{diam}(\tau)$. We suppose that $\bar\Omega$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are curved. Furthermore, we suppose that the triangulation is nondegenerate in the sense that $\max_{\tau\in\mathcal{T}_h}\frac{\mathrm{diam}(\tau)}{\rho_\tau} \le \kappa$, where the constant $\kappa > 0$ is independent of h and ρ_τ denotes the radius of the largest ball which is contained in $\bar{\tau}$. The discrete space is defined by

$$\mathcal{S}^h := \{v_h \in C^0(\bar{\Omega}) \mid v_h \text{ is a linear polynomial on each } \tau \in \mathcal{T}_h\}.$$

There exists an interpolation operator $\Pi^h : H^2(\Omega) \to \mathcal{S}^h$ such that

$$\|v - \Pi^h v\| + h\|\nabla(v - \Pi^h v)\| \leq ch^2 \|v\|_{H^2(\Omega)} \quad \forall\, v \in H^2(\Omega). \tag{3.1}$$

We are now in position to give a precise formulation of our numerical scheme. Let $\Delta t := \frac{T}{N}$ for an integer N and $t_m := m\Delta t$, $m = 0, \dots, N$. We denote by U^m, W^m the approximations to $u(\cdot, t_m)$ and $w(\cdot, t_m)$, respectively. Furthermore, let

$$\delta_t v^m := \frac{v^{m+1} - v^m}{\Delta t}.$$

In order to formulate a semi-implicit scheme requiring just the solution of linear equations we introduce the following form.

Stabilizing Anisotropic (SA) form,

$$\mathcal{B}(u; v, \eta) := \lambda \mathcal{B}_0(u; v, \eta) + \Delta t \mathcal{E}(u; v, \eta), \tag{3.2}$$

where

$$\mathcal{B}_0(u; v, \eta) := \int_\Omega \frac{\gamma(\nu(u))}{Q(u)} \nabla v \cdot \nabla w dx. \tag{3.3}$$

Remark 3.1. The purpose of the form $\mathcal{B}_0$ is to stabilize $\mathcal{A}$, which will be evaluated at the old time step. The second part in $\mathcal{B}$ is introduced in order to gain control on $\|U^m\|$ (see the proof of Lemma 3.4 below, in particular (3.14) and (3.15)) and the corresponding error in the convergence analysis.

Scheme 3.2. *We seek for each $m \in [1, N]$ a pair $\{U^m, W^m\} \in \mathcal{S}^h \times \mathcal{S}^h$ satisfying for $m \geq 0$*

$$(\delta_t U^m, \eta) + \mathcal{E}(U^m; W^{m+1}, \eta) = 0 \quad \forall\, \eta \in \mathcal{S}^h, \tag{3.4}$$

$$(W^{m+1}, \eta) - \mathcal{A}(U^m, \eta) - \Delta t \mathcal{B}(U^m; \delta_t U^m, \eta) = 0 \quad \forall\, \eta \in \mathcal{S}^h. \tag{3.5}$$

For simplicity we impose the initial condition,

$$U^0 := \Pi^h u_0. \tag{3.6}$$

The scheme does not require W^0. The constant λ is chosen to satisfy

$$\lambda \gamma_{\min} > \bar{\gamma}, \quad \text{where } \gamma_{\min} = \inf_{|p|=1} \gamma(p) > 0 \tag{3.7}$$

in order to ensure stability (see Lemma 3.4 below).

LEMMA 3.3 (properties of the SA form $\mathcal{B}$). *Suppose that $u, v \in H^{1,\infty}(\Omega)$. Then*

$$\mathcal{B}(u; v, v) \leq \left(\lambda \sup_{|p|=1} \gamma(p) + \Delta t\right) \mathcal{E}(u; v, v). \tag{3.8}$$

If in addition $|\nabla u| \leq K$ a.e. in Ω, then

$$\begin{aligned} &|\mathcal{B}(u; \eta_1, \eta_2) - \mathcal{B}(v; \eta_1, \eta_2)| \\ &\quad \leq C\|\nabla \eta_1\|_{L^\infty} \left(\int_\Omega |\nu(u) - \nu(v)||\nabla \eta_2| + \Delta t \int_\Omega |\nu(u) - \nu(v)||\nabla \eta_2| Q(v) \right). \end{aligned} \tag{3.9}$$

Proof. The inequality (3.8) follows immediately from (2.4). Next, if $|\nabla u| \le K$ a.e. in Ω, we deduce from (2.28) that

$$\begin{aligned}
&\left| \frac{\gamma(\nu(u))}{Q(u)} - \frac{\gamma(\nu(v))}{Q(v)} \right| \\
&\quad \le \frac{1}{Q(u)} \left| \int_0^1 \langle \gamma'(s\nu(u) + (1-s)\nu(v)) ds, \nu(u) - \nu(v) \rangle \right| + C \left| \frac{1}{Q(u)} - \frac{1}{Q(v)} \right| \\
&\quad \le C|\nu(u) - \nu(v)|.
\end{aligned}$$

Combining this inequality with (2.31) implies (3.9). ∎

3.2. Stability.

LEMMA 3.4. *Suppose that* (3.7) *holds. Then the unique discrete solution satisfies*

$$\max_{m \in [0,N]} \mathcal{I}_\gamma(U^m) + \Delta t \sum_{k=1}^{N} \mathcal{E}(U^{k-1}; W^k, W^k) \le C(\gamma, U^0), \tag{3.10}$$

$$\max_{m \in [0,N]} \|U^m\|^2 + \Delta t \sum_{k=1}^{N} \|W^k\|^2 \le C(\lambda, \gamma, U^0, T). \tag{3.11}$$

Proof. Taking $\eta = \Delta t W^{m+1}$ in (3.4), $\eta = \Delta t \delta_t U^m$ in (3.5) and adding yields

$$\begin{aligned}
&\Delta t \mathcal{E}(U^m; W^{m+1}, W^{m+1}) + \mathcal{A}(U^m, U^{m+1} - U^m) \\
&\qquad + (\Delta t)^2 \mathcal{B}(U^m; \delta_t U^m, \delta_t U^m) = 0.
\end{aligned} \tag{3.12}$$

Lemma 2.5 implies

$$\begin{aligned}
\mathcal{A}(U^m, U^{m+1} - U^m) &\ge I_\gamma(U^{m+1}) - I_\gamma(U^m) - \bar{\gamma} \int_\Omega |\nu(U^{m+1}) - \nu(U^m)|^2 Q(U^{m+1}) \\
&\ge I_\gamma(U^{m+1}) - I_\gamma(U^m) - (\Delta t)^2 \frac{\bar{\gamma}}{\gamma_{\min}} \mathcal{B}_0(U^m; \delta_t U^m, \delta_t U^m),
\end{aligned}$$

where we have used (2.21). Inserting the above inequality into (3.12) and recalling the definition of $\mathcal{B}$ we infer

$$\begin{aligned}
&\mathcal{I}_\gamma(U^{m+1}) - \mathcal{I}_\gamma(U^m) + \Delta t \mathcal{E}(U^m; W^{m+1}, W^{m+1}) \\
&\quad + \left(\lambda - \frac{\bar{\gamma}}{\gamma_{\min}} \right) (\Delta t)^2 \mathcal{B}_0(U^m; \delta_t U^m, \delta_t U^m) + (\Delta t)^3 \mathcal{E}(U^m, \delta_t U^m, \delta_t U^m) \le 0.
\end{aligned} \tag{3.13}$$

Summation over m yields (3.10) as well as

$$\begin{aligned}
&(\Delta t)^2 \sum_{m=0}^{N-1} \mathcal{B}_0(U^m; \delta_t U^m, \delta_t U^m) \\
&\qquad + (\Delta t)^3 \sum_{m=0}^{N-1} \mathcal{E}(U^m; \delta_t U^m, \delta_t U^m) \le C(\lambda, \gamma, U^0).
\end{aligned} \tag{3.14}$$

Next, using $\eta = \Delta t U^{m+1}$ in (3.4) we deduce

$$\begin{aligned}
&\frac{1}{2}\|U^{m+1}\|^2 - \frac{1}{2}\|U^m\|^2 + \frac{1}{2}\|U^{m+1} - U^m\|^2 = \Delta t \mathcal{E}(U^m; W^{m+1}, U^{m+1}) \\
&\quad \le \Delta t \mathcal{E}(U^m; W^{m+1}, W^{m+1})^{\frac{1}{2}} \mathcal{E}(U^m; U^{m+1}, U^{m+1})^{\frac{1}{2}} \\
&\quad \le \Delta t \mathcal{E}(U^m; W^{m+1}, W^{m+1})^{\frac{1}{2}} \left(\mathcal{E}(U^m; U^m, U^m)^{\frac{1}{2}} + \Delta t \mathcal{E}(U^m; \delta_t U^m, \delta_t U^m)^{\frac{1}{2}}\right) \\
&\quad \le \Delta t \mathcal{E}(U^m; W^{m+1}, W^{m+1}) + \Delta t \int_\Omega Q(U^m) + (\Delta t)^3 \mathcal{E}(U^m; \delta_t U^m, \delta_t U^m).
\end{aligned} \tag{3.15}$$

Finally, using $\eta = \Delta t W^{m+1}$ in (3.5) we obtain with the help of (2.4) and (3.8) that

$$\begin{aligned}
\Delta t \|W^{m+1}\|^2 &= \Delta t \mathcal{A}(U^m, W^{m+1}) + (\Delta t)^2 \mathcal{B}(U^m; \delta_t U^m, W^{m+1}) \\
&\le \Delta t \sup_{|p|=1} |\gamma'(p)| \left(\int_\Omega \frac{|\nabla W^{m+1}|^2}{Q(U^m)}\right)^{\frac{1}{2}} \left(\int_\Omega Q(U^m)\right)^{\frac{1}{2}} \qquad (3.16)\\
&\quad + (\Delta t)^2 \mathcal{B}(U^m; \delta_t U^m, \delta_t U^m)^{\frac{1}{2}} \mathcal{B}(U^m; W^{m+1}, W^{m+1})^{\frac{1}{2}} \\
&\le \Delta t \mathcal{E}(U^m; W^{m+1}, W^{m+1}) + C(\gamma) \Delta t \int_\Omega Q(U^m) \qquad (3.17)\\
&\quad + C(\Delta t)^2 \mathcal{B}(U^m; \delta_t U^m, \delta_t U^m).
\end{aligned}$$

Now (3.11) follows from summing (3.15), (3.16) over m, the inequality $\int_\Omega Q(U^m) \le C(\gamma) I_\gamma(U^m)$, and (3.10), (3.14). ◻

Remark 3.5. It follows in particular that

$$\max_{m \in [0,N]} \int_\Omega Q(U^m) \le C(\gamma, U^0). \tag{3.18}$$

3.3. Boundary conditions, domain perturbation, and quadrature. For Neumann boundary conditions it is sufficient for the union of the elements to contain Ω, provided exact quadrature is used. The above analysis can be easily extended to higher order elements. On the other hand, when using piecewise linear elements it is convenient to use a quadrature rule based on mass lumping for the L^2 inner products. The other integrals require just the measure of the regions of integration. In the case of Dirichlet boundary conditions it is necessary either to analyze the effect of domain perturbation in the case of linear finite elements with a polygonal interpolation of Ω or to analyze isoparametric approximations for higher order elements.

4. Error bounds.

We set

$$u^m := u(\cdot, t_m), \qquad w^m := w(\cdot, t_m), \qquad S^m := \delta_t u^m - \partial_t u(\cdot, t_{m+1}).$$

Then we have for the continuous problem the analogue of the discrete scheme,

$$(\delta_t u^m, \eta) + \mathcal{E}(u^{m+1}; w^{m+1}, \eta) = (S^m, \eta) \quad \forall\, \eta \in H^1(\Omega), \tag{4.1}$$

$$(w^m, \eta) - \mathcal{A}(u^m, \eta) = 0 \quad \forall\, \eta \in H^1(\Omega). \tag{4.2}$$

It is convenient to introduce the errors

$$e_u^m := u^m - U^m =: \rho_u^m + \theta_u^m, \qquad e_w^m := w^m - W^m := \rho_w^m + \theta_w^m,$$

where

$$\rho_u^m := u^m - \Pi^h u^m, \qquad \rho_w^m := w^m - \Pi^h w^m$$

are the interpolation errors. It is our goal to prove the following error bounds.

THEOREM 4.1. *Let* (u, w) *solve* (2.10)–(2.14) *and satisfy the regularity* $u \in H^{1,\infty}(0,T;H^{2,\infty}(\Omega))$, $u_{tt} \in L^\infty(0,T;H^{1,\infty}(\Omega))$, $w \in H^{1,\infty}(0,T;H^{2,\infty}(\Omega))$, $w_{tt} \in L^2(0,T;L^2(\Omega))$. *Suppose also that* (3.7) *holds. Then there exists* $\delta > 0$ *such that for* $0 < \Delta t \le \delta$

$$\max_{m\in[0,N]} \left(\|e_u^m\|^2 + \int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m) \right) + \Delta t \sum_{k=1}^{N} (\|e_w^k\|^2 + \mathcal{E}(U^{k-1}; e_w^k, e_w^k)) \le C(h^2 + (\Delta t)^2),$$

where C *and* δ *depend on* γ, Ω, T, λ *and the solution* u.

The rest of this section will be devoted to the proof of Theorem 4.1. Subtracting (3.4), (3.5) and (4.1), (4.2) yields, for all $\eta \in \mathcal{S}^h$, the error equations

$$(\delta_t e_u^m, \eta) + \mathcal{E}(u^{m+1}; w^{m+1}, \eta) - \mathcal{E}(U^m; W^{m+1}, \eta) = (S^m, \eta), \tag{4.3}$$

$$(e_w^{m+1}, \eta) - \mathcal{A}(u^m, \eta) + \mathcal{A}(U^m, \eta) + \Delta t \mathcal{B}(U^m; \delta_t U^m, \eta) = (w^{m+1} - w^m, \eta). \tag{4.4}$$

4.1. An a priori estimate in the energy norm. The first step is to emulate the energy bounds obtained for the continuous and discrete solutions by testing (4.3) and (4.4) with $e_w^{m+1} - \rho_w^{m+1} \in \mathcal{S}^h$ and $\delta_t e_u^m - \delta_t \rho_u^m \in \mathcal{S}^h$ yielding

$$\begin{aligned}(\delta_t e_u^m, e_w^{m+1}) &+ \mathcal{E}(u^{m+1}; w^{m+1}, e_w^{m+1}) - \mathcal{E}(U^m; W^{m+1}, e_w^{m+1}) \\ &= (\delta_t e_u^m, \rho_w^{m+1}) + \mathcal{E}(u^{m+1}; w^{m+1}, \rho_w^{m+1}) \\ &\quad - \mathcal{E}(U^m; W^{m+1}, \rho_w^{m+1}) + (S^m, e_w^{m+1} - \rho_w^{m+1}),\end{aligned} \tag{4.5}$$

$$\begin{aligned}(e_w^{m+1}, \delta_t e_u^m) &- \mathcal{A}(u^m, \delta_t e_u^m) + \mathcal{A}(U^m, \delta_t e_u^m) + \Delta t \mathcal{B}(U^m; \delta_t U^m, \delta_t e_u^m) \\ &= (e_w^{m+1}, \delta_t \rho_u^m) - \mathcal{A}(u^m, \delta_t \rho_u^m) + \mathcal{A}(U^m, \delta_t \rho_u^m) + \Delta t \mathcal{B}(U^m; \delta_t U^m, \delta_t \rho_u^m) \\ &\quad + \Delta t (\delta_t w^m, \delta_t e_u^m - \delta_t \rho_u^m).\end{aligned} \tag{4.6}$$

Combining these equations and multiplying by Δt yields

$$\begin{aligned}\Delta t(\mathcal{A}(u^m, \delta_t e_u^m) &- \mathcal{A}(U^m, \delta_t e_u^m)) \\ &+ \Delta t(\mathcal{E}(u^{m+1}; w^{m+1}, e_w^{m+1}) - \mathcal{E}(U^m; W^{m+1}, e_w^{m+1})) \\ &+ (\Delta t)^2 \mathcal{B}(U^m; \delta_t e_u^m, \delta_t e_u^m) = \Delta t(\mathcal{A}(u^m, \delta_t \rho_u^m) - \mathcal{A}(U^m, \delta_t \rho_u^m)) \\ &+ \Delta t(\mathcal{E}(u^{m+1}; w^{m+1}, \rho_w^{m+1}) - \mathcal{E}(U^m; W^{m+1}, \rho_w^{m+1})) \\ &+ \Delta t(S^m, e_w^{m+1} - \rho_w^{m+1}) - \Delta t(e_w^{m+1}, \delta_t \rho_u^m) \\ &+ \Delta t(\delta_t e_u^m, \rho_w^{m+1}) - (\Delta t)^2 (\delta_t w^m, \delta_t e_u^m - \delta_t \rho_u^m) \\ &+ (\Delta t)^2 (\mathcal{B}(U^m; \delta_t u^m, \delta_t e_u^m) - \mathcal{B}(U^m; \delta_t U^m, \delta_t \rho_u^m)) := \sum_{j=1}^{7} R_j^m.\end{aligned} \tag{4.7}$$

The proof of the error bounds is based on estimating the terms on both sides of the above equation. We begin with the left-hand side of (4.7) which we denote by L^m. First we recall the following lemma.

LEMMA 4.2. *Let*

$$\mathcal{D}^m := \int_\Omega (\gamma(\nu(U^m)) - \langle \gamma'(\nu(u^m)), \nu(U^m) \rangle) Q(U^m).$$

Then we have for $m \in [0, N-1]$ and small Δt

$$\Delta t(\mathcal{A}(u^m, \delta_t e_u^m) - \mathcal{A}(U^m, \delta_t e_u^m)) \geq \mathcal{D}^{m+1} - \mathcal{D}^m - (\bar{\gamma} + C\Delta t) \int_\Omega \frac{|\nabla(e_u^{m+1} - e_u^m)|^2}{Q(U^m)} - C\Delta t \left((\Delta t)^2 + \int_\Omega |\nu(u^{m+1}) - \nu(U^{m+1})|^2 Q(U^{m+1}) \right).$$

Proof. See [11, Lemma 4.2]. ∎

Lemma 4.2 and the definition of $\mathcal{B}_0$ now imply

$$\begin{aligned} &\Delta t(\mathcal{A}(u^m, \delta_t e_u^m) - \mathcal{A}(U^m, \delta_t e_u^m)) \\ &\quad \geq \mathcal{D}^{m+1} - \mathcal{D}^m - (\Delta t)^2 \left(\frac{\bar{\gamma}}{\gamma_{\min}} + C\Delta t \right) \mathcal{B}_0(U^m; \delta_t e_u^m, \delta_t e_u^m) \\ &\quad - C\Delta t \left((\Delta t)^2 + \int_\Omega |\nu(u^{m+1}) - \nu(U^{m+1})|^2 Q(U^{m+1}) \right). \end{aligned} \tag{4.8}$$

Next we examine

$$\begin{aligned} &\Delta t(\mathcal{E}(u^{m+1}; w^{m+1}, e_w^{m+1}) - \mathcal{E}(U^m; W^{m+1}, e_w^{m+1})) \\ &\quad = \Delta t \mathcal{E}(U^m; e_w^{m+1}, e_w^{m+1}) + \Delta t(\mathcal{E}(u^{m+1}; w^{m+1}, e_w^{m+1}) - \mathcal{E}(u^m; w^{m+1}, e_w^{m+1})) \\ &\qquad + \Delta t(\mathcal{E}(u^m; w^{m+1}, e_w^{m+1}) - \mathcal{E}(U^m; w^{m+1}, e_w^{m+1})) \\ &\quad =: \alpha_1^m + \alpha_2^m + \alpha_3^m. \end{aligned}$$

We infer from (3.18) and (2.4) that

$$\begin{aligned} |\alpha_2^m| &\leq C(\Delta t)^2 \|\nabla w^{m+1}\|_{L^\infty} \int_\Omega |\nabla e_w^{m+1}| \leq C(\Delta t)^2 \left(\int_\Omega Q(U^m) \right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla e_w^{m+1}|^2}{Q(U^m)} \right)^{\frac{1}{2}} \\ &\leq \epsilon \Delta t \mathcal{E}(U^m, e_w^{m+1}, e_w^{m+1}) + \frac{C}{\epsilon} (\Delta t)^3. \end{aligned}$$

Furthermore, (2.32) yields

$$|\alpha_3^m| \leq \epsilon \Delta t \mathcal{E}(U^m, e_w^{m+1}, e_w^{m+1}) + \frac{C}{\epsilon} \|\nabla w^{m+1}\|_{L^\infty}^2 \int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m).$$

Combining (4.8) and the estimates for α_2^m, α_3^m we derive

$$\begin{aligned} L^m &\geq \mathcal{D}^{m+1} - \mathcal{D}^m + (1 - 2\epsilon)\Delta t \mathcal{E}(U^m; e_w^{m+1}, e_w^{m+1}) \\ &\quad + (\Delta t)^2 \left(\lambda - \frac{\bar{\gamma}}{\gamma_{\min}} - C\Delta t \right) \mathcal{B}_0(U^m; \delta_t e_u^m, \delta_t e_u^m) + (\Delta t)^3 \mathcal{E}(U^m; \delta_t e_u^m, \delta_t e_u^m) \\ &\quad - \frac{C}{\epsilon} \Delta t \left((\Delta t)^2 + \int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m) \right. \\ &\qquad \left. + \int_\Omega |\nu(u^{m+1}) - \nu(U^{m+1})|^2 Q(U^{m+1}) \right). \end{aligned} \tag{4.9}$$

4.2. L^2-estimates. In order to proceed and estimate the terms R_j^m on the right-hand side of (4.7), we need to derive bounds on the L^2-norms of e_w^{k+1} and e_u^{k+1}.

LEMMA 4.3. *We have for $m \in [0, N-1]$*

$$\|e_w^{m+1}\|^2 \le \mathcal{E}(U^m; e_w^{m+1}, e_w^{m+1}) + C(\Delta t)^2 \mathcal{B}(U^m; \delta_t e_u^m, \delta_t e_u^m) + C \int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m) + C(h^2 + (\Delta t)^2).$$

Proof. Inserting $\eta = e_w^{m+1} - \rho_w^{m+1}$ into (4.4) and using (2.26) we infer

$$\begin{aligned}
&\|e_w^{m+1}\|^2 \\
&= (e_w^{m+1}, e_w^{m+1} - \rho_w^{m+1}) + (e_w^{m+1}, \rho_w^{m+1}) \\
&= \mathcal{A}(u^m, e_w^{m+1} - \rho_w^{m+1}) - \mathcal{A}(U^m, e_w^{m+1} - \rho_w^{m+1}) \\
&\quad - \Delta t \mathcal{B}(U^m; \delta_t U^m, e_w^{m+1} - \rho_w^{m+1}) + \Delta t(\delta_t w^m, e_w^{m+1} - \rho_w^{m+1}) \\
&\quad + (e_w^{m+1}, \rho_w^{m+1}) \le C \int_\Omega |\nu(u^m) - \nu(U^m)|(|\nabla e_w^{m+1}| + |\nabla \rho_w^{m+1}|) \\
&\quad + \Delta t |\mathcal{B}(U^m; \delta_t U^m, e_w^{m+1} - \rho_w^{m+1})| + C\Delta t(\|e_w^{m+1}\| + \|\rho_w^{m+1}\|) \\
&\quad + \|e_w^{m+1}\| \|\rho_w^{m+1}\| \le \frac{1}{2}\|e_w^{m+1}\|^2 + \Delta t |\mathcal{B}(U^m; \delta_t U^m, e_w^{m+1} - \rho_w^{m+1})| \\
&\quad + C((\Delta t)^2 + h^2) + \frac{1}{4}\int_\Omega \frac{|\nabla e_w^{m+1}|^2}{Q(U^m)} + C \int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m).
\end{aligned}$$

It remains to bound the term involving $\mathcal{B}$. Clearly,

$$\begin{aligned}
&|\mathcal{B}(U^m; \delta_t U^m, e_w^{m+1})| \\
&\quad \le \mathcal{B}(U^m; \delta_t U^m, \delta_t U^m)^{\frac{1}{2}} \mathcal{B}(U^m; e_w^{m+1}, e_w^{m+1})^{\frac{1}{2}} \\
&\quad \le \left(\mathcal{B}(U^m; \delta_t u^m, \delta_t u^m)^{\frac{1}{2}} + \mathcal{B}(U^m; \delta_t e_u^m, \delta_t e_u^m)^{\frac{1}{2}}\right) \mathcal{B}(U^m; e_w^{m+1}, e_w^{m+1})^{\frac{1}{2}} \\
&\quad \le C\left(\left(\int_\Omega Q(U^m)\right)^{\frac{1}{2}} + \mathcal{B}(U^m; \delta_t e_u^m, \delta_t e_u^m)^{\frac{1}{2}}\right) \mathcal{E}(U^m; e_w^{m+1}, e_w^{m+1})^{\frac{1}{2}},
\end{aligned}$$

by (3.8). Recalling (3.18) we deduce

$$\begin{aligned}
&\Delta t |\mathcal{B}(U^m; \delta_t U^m, e_w^{m+1})| \\
&\quad \le \frac{1}{4}\mathcal{E}(U^m; e_w^{m+1}, e_w^{m+1}) + C((\Delta t)^2 + (\Delta t)^2 \mathcal{B}(U^m; \delta_t e_u^m, \delta_t e_u^m)).
\end{aligned}$$

Similarly,

$$\Delta t |\mathcal{B}(U^m; \delta_t U^m, \rho_w^{m+1})| \le C((\Delta t)^2 + h^2) + C(\Delta t)^2 \mathcal{B}(U^m; \delta_t e_u^m, \delta_t e_u^m).$$

If we insert these inequalities into the estimate for $\|e_w^{m+1}\|$ and use (2.4) we arrive at the desired bound. ∎

LEMMA 4.4. *We have for $0 \le m \le N$*

$$\max_{k \in [0,m]} \|e_u^k\|^2 \le C\left(\Delta t \sum_{k=0}^{m-1} \mathcal{E}(U^k; e_w^{k+1}, e_w^{k+1}) + (\Delta t)^3 \sum_{k=0}^{m-1} \mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k)\right) + C((\Delta t)^2 + h^2) + C\Delta t \sum_{k=0}^{m-1} \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k).$$

Proof. Clearly,

$$\begin{aligned}
&\frac{1}{2}\|e_u^{k+1}\|^2 - \frac{1}{2}\|e_u^k\|^2 + \frac{1}{2}\|e_u^{k+1} - e_u^k\|^2 \\
&\quad = \Delta t(\delta_t e_u^k, e_u^{k+1}) = \Delta t(\delta_t e_u^k, \theta_u^{k+1}) + \Delta t(\delta_t e_u^k, \rho_u^{k+1}) \\
&\quad = \Delta t(\mathcal{E}(U^k; W^{k+1}, \theta_u^{k+1}) - \mathcal{E}(u^{k+1}; w^{k+1}, \theta_u^{k+1})) \\
&\qquad + \Delta t(S^k, \theta_u^{k+1}) + \Delta t(\delta_t e_u^k, \rho_u^{k+1}),
\end{aligned} \tag{4.10}$$

where the last inequality follows from (4.3) with the choice $\eta = \Delta t\theta_u^{k+1}$. To begin,

$$\begin{aligned}
&|\mathcal{E}(U^k; W^{k+1}, \theta_u^{k+1}) - \mathcal{E}(u^{k+1}; w^{k+1}, \theta_u^{k+1})| \\
&\quad \le |\mathcal{E}(U^k; e_w^{k+1}, \theta_u^{k+1})| + |\mathcal{E}(U^k; w^{k+1}, \theta_u^{k+1}) - \mathcal{E}(u^k; w^{k+1}, \theta_u^{k+1})| \\
&\qquad + |\mathcal{E}(u^k; w^{k+1}, \theta_u^{k+1}) - \mathcal{E}(u^{k+1}; w^{k+1}, \theta_u^{k+1})| \\
&\quad = I + II + III.
\end{aligned}$$

Before we estimate these terms we first note that (2.30) and (3.18) imply

$$\begin{aligned}
&\mathcal{E}(U^k; \theta_u^{k+1}, \theta_u^{k+1}) \\
&\quad \le 2\mathcal{E}(U^k; e_u^{k+1}, e_u^{k+1}) + 2\mathcal{E}(U^k; \rho_u^{k+1}, \rho_u^{k+1}) \\
&\quad \le 4\mathcal{E}(U^k; e_u^k, e_u^k) + 4(\Delta t)^2\mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + C\|\nabla\rho_u^{k+1}\|_{L^\infty}^2 \int_\Omega Q(U^k) \\
&\quad \le C\int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k) + 4(\Delta t)^2\mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + Ch^2.
\end{aligned} \tag{4.11}$$

We then infer from (2.29) and (4.11)

$$\begin{aligned}
I &\le \mathcal{E}(U^k; e_w^{k+1}, e_w^{k+1})^{\frac{1}{2}}\mathcal{E}(U^k; \theta_u^{k+1}, \theta_u^{k+1})^{\frac{1}{2}} \le \mathcal{E}(U^k; e_w^{k+1}, e_w^{k+1}) \\
&\quad + C\bigg((\Delta t)^2\mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + h^2 + \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k)\bigg).
\end{aligned}$$

Next, (2.32) together with (4.11) implies

$$\begin{aligned}
II &\le \mathcal{E}(U^k; \theta_u^{k+1}, \theta_u^{k+1}) + C\|\nabla w^{k+1}\|_{L^\infty}^2 \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k) \\
&\le C\left((\Delta t)^2\mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + h^2 + \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k)\right),
\end{aligned}$$

as well as

$$\begin{aligned}
III &\le \mathcal{E}(U^k; \theta_u^{k+1}, \theta_u^{k+1}) + C\|\nabla w^{k+1}\|_{L^\infty}^2 \int_\Omega |\nu(u^{k+1}) - \nu(u^k)|^2 Q(u^k) \\
&\le C\left((\Delta t)^2\mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + (\Delta t)^2 + h^2 + \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k)\right).
\end{aligned}$$

Collecting the above estimates we derive

$$\begin{aligned}
&\Delta t|\mathcal{E}(U^k; W^{k+1}, \theta_u^{k+1}) - \mathcal{E}(u^{k+1}; w^{k+1}, \theta_u^{k+1})| \le \Delta t\mathcal{E}(U^k; e_w^{k+1}, e_w^{k+1}) \\
&\quad + C\Delta t\left((\Delta t)^2\mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + (\Delta t)^2 + h^2 + \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k)\right).
\end{aligned}$$

Next,

$$\begin{aligned}\Delta t|(S^k,\theta_u^{k+1})| &\le C(\Delta t)^2(\|e_u^k\| + \|e_u^{k+1}-e_u^k\| + \|\rho_u^{k+1}\|)\\ &\le \frac{1}{4}\|e_u^{k+1}-e_u^k\|^2 + C\Delta t\|e_u^k\|^2 + C\Delta t((\Delta t)^2+h^4).\end{aligned}$$

If we insert the above estimates into (4.10), sum from $k=0$ to $m-1$, and rearrange terms, we obtain

$$\begin{aligned}\frac{1}{2}\|e_u^m\|^2 \le{}& \frac{1}{2}\|e_u^0\|^2 + \Delta t\sum_{k=0}^{m-1}\mathcal{E}(U^k;e_w^{k+1},e_w^{k+1}) + C(\Delta t)^3\sum_{k=0}^{m-1}\mathcal{E}(U^k;\delta_t e_u^k,\delta_t e_u^k)\\ &+ C((\Delta t)^2+h^2) + C\Delta t\sum_{k=0}^{m-1}\int_\Omega|\nu(u^k)-\nu(U^k)|^2Q(U^k)\\ &+ C\Delta t\sum_{k=0}^{m-1}\|e_u^k\|^2 + \Delta t\sum_{k=0}^{m-1}(\delta_t e_u^k,\rho_u^{k+1}).\end{aligned}$$

Integrating by parts discretely in time we infer

$$\left|\Delta t\sum_{k=0}^{m-1}(\delta_t e_u^k,\rho_u^{k+1})\right| = \left|-\Delta t\sum_{k=0}^{m-1}(e_u^k,\delta_t\rho_u^k) + (e_u^m,\rho_u^m) - (e_u^0,\rho_u^0)\right| \le Ch^2,$$

since $\max_{k\in[0,N]}\|e_u^k\|^2 \le C$ by Lemma 3.4. Thus,

$$\begin{aligned}\|e_u^m\|^2 \le{}& 2\Delta t\sum_{k=0}^{m-1}\mathcal{E}(U^k;e_w^{k+1},e_w^{k+1}) + C(\Delta t)^3\sum_{k=0}^{m-1}\mathcal{E}(U^k;\delta_t e_u^k,\delta_t e_u^k) + C((\Delta t)^2+h^2)\\ &+ C\Delta t\sum_{k=0}^{m-1}\int_\Omega|\nu(u^k)-\nu(U^k)|^2Q(U^k) + C\Delta t\sum_{k=0}^{m-1}\|e_u^k\|^2.\end{aligned}$$

The result now follows with the help of a discrete Gronwall argument. □

4.3. Estimating the right-hand side of (4.7). Invoking (2.26) we obtain

$$\begin{aligned}|R_1^k| &= \Delta t|\mathcal{A}(u^k,\delta_t\rho_u^k) - \mathcal{A}(U^k,\delta_t\rho_u^k)|\\ &\le C\Delta t\int_\Omega|\nu(u^k)-\nu(U^k)||\nabla\delta_t\rho_u^k| \le C\Delta t h\left(\int_\Omega|\nu(u^k)-\nu(U^k)|^2\right)^{\frac{1}{2}}\\ &\le C\Delta t h^2 + C\Delta t\int_\Omega|\nu(u^k)-\nu(U^k)|^2Q(U^k).\end{aligned}\tag{4.12}$$

Lemma 2.6 and (3.18) imply

$$
\begin{aligned}
|R_2^k| \le{}& \Delta t|\mathcal{E}(u^{k+1};w^{k+1},\rho_w^{k+1}) - \mathcal{E}(u^k;w^{k+1},\rho_w^{k+1})| \\
&+ \Delta t|\mathcal{E}(u^k;w^{k+1},\rho_w^{k+1}) - \mathcal{E}(U^k;w^{k+1},\rho_w^{k+1})| \\
&+ \Delta t|\mathcal{E}(U^k;e_w^{k+1},\rho_w^{k+1})| \le C\Delta t\|\nabla w^{k+1}\|_{L^\infty} \\
&\times\biggl(\int_\Omega |\nu(u^{k+1}) - \nu(u^k)||\nabla \rho_w^{k+1}|Q(u^k) \\
&\quad + \int_\Omega |\nu(u^k) - \nu(U^k)||\nabla \rho_w^{k+1}|Q(U^k)\biggr) \\
&+ \Delta t\mathcal{E}(U^k;e_w^{k+1},e_w^{k+1})^{\frac12}\mathcal{E}(U^k;\rho_w^{k+1},\rho_w^{k+1})^{\frac12} \\
\le{}& \epsilon\Delta t\mathcal{E}(U^k;e_w^{k+1},e_w^{k+1}) + \frac{C}{\epsilon}\Delta t((\Delta t)^2 + h^2) \\
&+ C\Delta t\int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k).
\end{aligned}
\tag{4.13}
$$

Next, Lemma 4.3 gives

$$
\begin{aligned}
|R_3^k + R_4^k| &\le C(\Delta t)^2(\|e_w^{k+1}\| + \|\rho_w^{k+1}\|) + \Delta t\|e_w^{k+1}\|\|\delta_t \rho_u^k\| \\
&\le \epsilon\Delta t\|e_w^{k+1}\|^2 + \frac{C}{\epsilon}\Delta t((\Delta t)^2 + h^4) \\
&\le \epsilon\Delta t\mathcal{E}(U^k;e_w^{k+1},e_w^{k+1}) + \epsilon(\Delta t)^3\mathcal{B}(U^k;\delta_t e_u^k,\delta_t e_u^k) \\
&\quad + \epsilon\Delta t\int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k) + \frac{C}{\epsilon}\Delta t((\Delta t)^2 + h^2).
\end{aligned}
\tag{4.14}
$$

Integrating by parts discretely in time yields

$$
\begin{aligned}
\left|\sum_{k=0}^{m-1} R_5^k\right| &= \left|-\Delta t\sum_{k=0}^{m-1}(e_u^k,\delta_t\rho_w^k) + (e_u^m,\rho_w^m) - (e_u^0,\rho_w^0)\right| \\
&\le \epsilon \max_{k\in[0,m]}\|e_u^k\|^2 + \frac{C}{\epsilon}h^4.
\end{aligned}
\tag{4.15}
$$

Similarly,

$$
\begin{aligned}
\left|\sum_{k=0}^{m-1} R_6^k\right| &\le \left|-(\Delta t)^2\sum_{k=0}^{m-1}(\delta_t w^k,\delta_t e_u^k)\right| + (\Delta t)^2\left|\sum_{k=0}^{m-1}(\delta_t w^k,\delta_t\rho_u^k)\right| \\
&\le (\Delta t)^2\left|\sum_{k=1}^{m-1}\left(\frac{w^{k+1} - 2w^k + w^{k-1}}{(\Delta t)^2}, e_u^k\right)\right. \\
&\qquad \left. - \Delta t(\delta_t w^{m-1},e_u^m) + \Delta t(\delta_t w^0,e_u^0)\right| + Ch^2\Delta t \\
&\le C\Delta t\max_{k\in[0,m]}\|e_u^k\| + Ch^2\Delta t \le \epsilon\max_{k\in[0,m]}\|e_u^k\|^2 + \frac{C}{\epsilon}((\Delta t)^2 + h^4).
\end{aligned}
\tag{4.16}
$$

Finally, let us write

$$
\begin{aligned}
R_7^k &= (\Delta t)^2(\mathcal{B}(U^k;\delta_t u^k,\delta_t e_u^k) - \mathcal{B}(U^k;\delta_t u^k,\delta_t\rho_u^k) + \mathcal{B}(U^k;\delta_t e_u^k,\delta_t\rho_u^k)) \\
&= I + II + III.
\end{aligned}
$$

In view of the definition of $\mathcal{B}$ we have

$$\begin{aligned} I &= (\Delta t)^2 \lambda \int_\Omega \frac{\gamma(\nu(u^k))}{Q(u^k)} \nabla \delta_t u^k \cdot \nabla \delta_t e_u^k \\ &\quad + (\Delta t)^2 \lambda \int_\Omega \left(\frac{\gamma(\nu(U^k))}{Q(U^k)} - \frac{\gamma(\nu(u^k))}{Q(u^k)} \right) \nabla \delta_t u^k \cdot \nabla \delta_t e_u^k \\ &\quad + (\Delta t)^3 \mathcal{E}(U^k; \delta_t u^k, \delta_t e_u^k) = (\Delta t)^2 \lambda (G^k, \nabla \delta_t e_u^k) + I_2 + I_3, \end{aligned}$$

where we have written $G^k := \frac{\gamma(\nu(u^k))}{Q(u^k)} \nabla \delta_t u^k$. We infer from (2.28) and (2.4) that

$$\begin{aligned} |I_2| &\le C(\Delta t)^2 \int_\Omega |\nu(u^k) - \nu(U^k)| |\nabla \delta_t e_u^k| \\ &\le \epsilon (\Delta t)^3 \int_\Omega \frac{|\nabla \delta_t e_u^k|^2}{Q(U^k)} + \frac{C}{\epsilon} \Delta t \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k) \\ &\le \epsilon (\Delta t)^3 \mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + \frac{C}{\epsilon} \Delta t \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k). \end{aligned}$$

Furthermore, (2.29) and (3.18) yield

$$|I_3| \le \epsilon (\Delta t)^3 \mathcal{E}(U^k; \delta_t e_u^k, \delta_t e_u^k) + \frac{C}{\epsilon} (\Delta t)^3.$$

Observing that $\mathcal{B}(U^k; \delta_t \rho_u^k, \delta_t \rho_u^k) \le C h^2$ we finally have

$$\begin{aligned} |II| &\le (\Delta t)^2 \mathcal{B}(U^k; \delta_t u^k, \delta_t u^k)^{\frac{1}{2}} \mathcal{B}(U^k; \delta_t \rho_u^k, \delta_t \rho_u^k)^{\frac{1}{2}} \le C \Delta t ((\Delta t)^2 + h^2), \\ |III| &\le (\Delta t)^2 \mathcal{B}(U^k; \delta_t e_u^k, \delta_t e_u^k)^{\frac{1}{2}} \mathcal{B}(U^k; \delta_t \rho_u^k, \delta_t \rho_u^k)^{\frac{1}{2}} \\ &\le \epsilon (\Delta t)^2 \mathcal{B}(U^k; \delta_t e_u^k, \delta_t e_u^k) + \frac{C}{\epsilon} \Delta t ((\Delta t)^2 + h^2). \end{aligned}$$

Summing the above estimates, integrating the first term in I by parts in time, and taking into account the estimate (which follows from (2.21) and (2.23))

$$\|\nabla e_u^k\|_{L^1} \le \left(\int_\Omega \frac{|\nabla e_u^k|^2}{Q(U^k)} \right)^{\frac{1}{2}} \left(\int_\Omega Q(U^k) \right)^{\frac{1}{2}} \le C \left(\int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k) \right)^{\frac{1}{2}},$$

we derive

$$\begin{aligned} \left| \sum_{k=0}^{m-1} R_7^k \right| &\le \lambda \left| -(\Delta t)^2 \sum_{k=0}^{m-1} (\delta_t G^k, \nabla e_u^k) + \Delta t (G^m, \nabla e_u^m) - \Delta t (G^0, \nabla e_u^0) \right| \\ &\quad + \frac{C}{\epsilon} ((\Delta t)^2 + h^2) + \epsilon (\Delta t)^2 \sum_{k=0}^{m-1} \mathcal{B}(U^k, \delta_t e_u^k, \delta_t e_u^k) \\ &\quad + \frac{C}{\epsilon} \Delta t \sum_{k=0}^{m-1} \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k) \\ &\le \epsilon \int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m) + \epsilon (\Delta t)^2 \sum_{k=0}^{m-1} \mathcal{B}(U^k, \delta_t e_u^k, \delta_t e_u^k) \\ &\quad + \frac{C}{\epsilon} ((\Delta t)^2 + h^2) + \frac{C}{\epsilon} \Delta t \sum_{k=0}^{m-1} \int_\Omega |\nu(u^k) - \nu(U^k)|^2 Q(U^k). \end{aligned} \tag{4.17}$$

Collecting (4.12)–(4.17) and recalling Lemma 2.8 finally yields

$$
\begin{aligned}
\left|\sum_{k=0}^{m-1}\sum_{j=1}^{7} R_j^k\right| \le\ & \epsilon\Delta t\sum_{k=0}^{m-1}\mathcal{E}(U^k; e_w^{k+1}, e_w^{k+1}) \\
&+\epsilon(\Delta t)^2\sum_{k=0}^{m-1}\mathcal{B}_0(U^k;\delta_t e_u^k,\delta_t e_u^k)+\epsilon\max_{k\in[0,N]}\|e_u^k\|^2 \\
&+\epsilon(\Delta t)^3\sum_{k=0}^{m-1}\mathcal{E}(U^k;\delta_t e_u^k,\delta_t e_u^k)+\frac{C}{\epsilon}((\Delta t)^2+h^2) \\
&+C\Delta t\sum_{k=0}^{m}\mathcal{D}^k+\epsilon\mathcal{D}^m.
\end{aligned} \tag{4.18}
$$

4.4. Completion of the proof of the error bound. We are now in position to complete the proof of the error estimate. Starting from the relation $\sum_{k=0}^{m-1} L^k = \sum_{k=0}^{m-1}\sum_{j=1}^{7} R_j^k$ and using (4.9) together with (4.18) and Lemma 4.4 we deduce

$$
\begin{aligned}
(1-\epsilon)\mathcal{D}^m &+ (\Delta t)^2\left(\lambda-\frac{\bar\gamma}{\gamma_{\min}}-\epsilon-C\Delta t\right)\sum_{k=0}^{m-1}\mathcal{B}_0(U^k;\delta_t e_u^k,\delta_t e_u^k) \\
&+(1-C\epsilon)\Delta t\sum_{k=0}^{m-1}\mathcal{E}(U^k;e_w^{k+1},e_w^{k+1})+(1-C\epsilon)(\Delta t)^3\sum_{k=0}^{m-1}\mathcal{E}(U^k;\delta_t e_u^k,\delta_t e_u^k) \\
&\le \mathcal{D}^0+\frac{C}{\epsilon}((\Delta t)^2+h^2)+\frac{C}{\epsilon}\Delta t\sum_{k=0}^{m}\mathcal{D}^k.
\end{aligned}
$$

It follows from (2.7) that $\mathcal{D}^0=\int_\Omega(\gamma(\nu(U^0))-\gamma(\nu(u^0))-\langle\gamma'(\nu(u^0)),(\nu(U^0)-\nu(u^0))\rangle)Q(U^0)$ so that by Taylor expansion and (2.28) $\mathcal{D}^0\le Ch^2$. After choosing ϵ and Δt sufficiently small we obtain

$$
\begin{aligned}
\mathcal{D}^m+\frac{\Delta t}{2}\sum_{k=1}^{m}\mathcal{E}(U^k,e_w^{k+1},e_w^{k+1})+c_0(\Delta t)^2\sum_{k=0}^{m-1}\mathcal{B}(U^k;\delta_t e_u^k,\delta_t e_u^k) \\
\le C((\Delta t)^2+h^2)+C\Delta t\sum_{k=0}^{m-1}\mathcal{D}^k.
\end{aligned}
$$

Gronwall's lemma together with Lemma 2.8 implies that

$$
\begin{aligned}
\max_{m\in[0,N]}\int_\Omega|\nu(u^m)-\nu(U^m)|^2Q(U^m)+\Delta t\sum_{k=1}^{N}\mathcal{E}(U^{k-1},e_w^k,e_w^k) \\
+(\Delta t)^2\sum_{k=0}^{N-1}\mathcal{B}(U^k;\delta_t e_u^k,\delta_t e_u^k)\le C((\Delta t)^2+h^2)
\end{aligned}
$$

and the remainder of the proof of Theorem 4.1 now follows from Lemmas 4.3 and 4.4.

5. Numerical results.

5.1. The algebraic problem. Let $\{\chi_j\}$ denote the usual nodal basis functions for $\mathcal{S}^h$. Set

$$M_{i,j} = (\chi_i, \chi_j), \qquad E^m_{i,j} = \mathcal{E}(U^m; \chi_i, \chi_j), \qquad B^m_{i,j} = \mathcal{B}(U^m; \chi_i, \chi_j)$$

and

$$F^m_j = -\mathcal{A}(U^m, \chi_j) + \mathcal{B}(U^m; U^m, \chi_j).$$

It follows that the nodal values $\mathbf{U}^{m+1}$, $\mathbf{W}^{m+1}$ solve the linear algebraic system

$$\frac{1}{\Delta t} M\mathbf{U}^{m+1} + E^m\mathbf{W}^{m+1} = \frac{1}{\Delta t} M\mathbf{U}^m,$$
$$B^m\mathbf{U}^{m+1} - M\mathbf{W}^{m+1} = \mathbf{F}^m.$$

Note that the structure of this system is of the same form as that arising in discretizations of the Cahn–Hilliard equation. Eliminating $\mathbf{W}^{m+1}$ by inverting the mass matrix in the second equation leads to the "fourth order" system

$$\frac{1}{\Delta t} M\mathbf{U}^{m+1} + E^m M^{-1} B^m \mathbf{U}^{m+1} = \frac{1}{\Delta t} M\mathbf{U}^m + E^m M^{-1}\mathbf{F}^m. \tag{5.1}$$

In our practical computations we have used mass lumping, so that M becomes a diagonal matrix. Although the system is unsymmetric, both the biconjugate gradient (BICG) and conjugate gradient (CG) methods were used to solve the linear equations. Remarkably, it was discovered that CG converged.

5.2. Convergence tests. We measured the actual error in different norms for several quantities for test problems, for which we know the continuous solutions. For this we have to extend our method to include right-hand sides f and g as indicated in (1.5). The tables contain the errors for the graph $u = u(x,t)$,

$$E_{\infty,2,u} = \max_{m\in[0,N]} \|u^m - U^m\|,$$
$$E_{\infty,2,\nu} = \max_{m\in[0,N]} \left(\int_\Omega |\nu(u^m) - \nu(U^m)|^2 Q(U^m)\right)^{\frac{1}{2}},$$

and for the curvature $w = w(x,t)$,

$$E_{2,2,w} = \left(\Delta t \sum_{m=0}^{N} \|w^m - W^m\|^2\right)^{\frac{1}{2}},$$
$$E_{2,\mathcal{E},\nabla w} = \left(\Delta t \sum_{m=0}^{M-1} \mathcal{E}(U^m; w^m - W^m, w^m - W^m)\right)^{\frac{1}{2}}.$$

These are the errors which were estimated in Theorem 4.1. Additionally we provide the errors

$$E_{\infty,\infty,u} = \max_{m\in[0,N]} \|u^m - U^m\|_{L^\infty}, \qquad E_{\infty,2,\nabla u} = \max_{m\in[0,N]} \|\nabla u^m - \nabla U^m\|.$$

TABLE 5.1
Errors for the isotropic test problem with $\Delta t = 0.1h$.

h	$E_{\infty,2,u}$	*eoc*	$E_{\infty,2,\nu}$	*eoc*	$E_{2,2,w}$	*eoc*	$E_{2,\mathcal{E},\nabla w}$	*eoc*
1.0	8.495	-	0.4538	-	0.2264	-	5.534	-
0.7368	3.299	3.10	0.1702	3.21	0.6294	−3.35	2.965	2.04
0.4203	0.6255	2.96	0.06580	1.69	0.2343	1.76	1.097	1.77
0.2219	0.1564	2.17	0.03241	1.11	0.06291	2.06	0.4664	1.34
0.1137	0.04360	1.91	0.01622	1.04	0.01597	2.05	0.2234	1.10
0.05754	0.01306	1.77	0.008113	1.02	0.003942	2.05	0.1109	1.03

We also measure the error

$$E_{2,2,\nabla w} = \left(\Delta t \sum_{m=0}^{M-1} \int_\Omega \frac{|\nabla w^m - \nabla W^m|^2}{Q(U^m)} \right)^{\frac{1}{2}},$$

which is bounded from above by $E_{2,\mathcal{E},\nabla w}$. The error in the normal velocity is given by

$$E_{2,2,V} = \left(\Delta t \sum_{m=1}^{N} \int_\Omega (V(u^m) - V(U^m))^2 Q(U^m) \right)^{\frac{1}{2}},$$

where

$$V(u^m) = -\frac{u_t(\cdot, t_m)}{Q(u^m)}, \qquad V(U^m) = -\frac{U^m - U^{m-1}}{\Delta t\, Q(U^m)}.$$

Between two spatial discretization levels with grid sizes h_1 and h_2 we compute the experimental order of convergence

$$eoc(h_1, h_2) = \log \frac{E(h_1)}{E(h_2)} \left(\log \frac{h_1}{h_2} \right)^{-1}$$

for the errors $E(h_1)$ and $E(h_2)$ for each of the error norms.

For isotropic surface diffusion we used the function

$$u(x,t) = \frac{1}{2}\cos(t) \left(1 + |x|^2 - \frac{3}{4}|x|^4 + \frac{1}{6}|x|^6 \right)$$

as continuous solution on the domain $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ and on the time interval $[0,T] = [0,1]$. We calculated the right-hand side g from the equation

$$g = V - \Delta_\Gamma \mathcal{H}_\gamma,$$

and used this function as a right-hand side in our algorithm to compute U^m and W^m. We have chosen $\lambda = 1$. In Tables 5.1 and 5.2 we show the results for the time step size $\Delta t = 0.1\,h$ and Tables 5.3 and 5.4 contain the results for $\Delta t = h^2$. The results confirm the theoretical estimates from Theorem 4.1. Obviously the errors $E_{\infty,2,\nu}$ and $E_{\infty,2,\nabla u}$ as well as the errors $E_{2,\mathcal{E},\nabla w}$ and $E_{2,2,\nabla w}$ exhibit the same orders of convergence.

The anisotropic case was tested, see Tables 5.5 and 5.6, with the exact solution

$$u(x,t) = \sqrt{1 - 4t - 4x_1^2 - x_2^2}$$

Table 5.2
Errors for the isotropic test problem with $\Delta t = 0.1h$.

h	$E_{\infty,\infty,u}$	eoc	$E_{2,2,V}$	eoc	$E_{\infty,2,\nabla u}$	eoc	$E_{2,2,\nabla w}$	eoc
1.0	5.027	-	9.113	-	0.4676	-	5.529	-
0.7368	1.848	3.28	3.548	3.09	0.1767	3.19	2.952	2.06
0.4203	0.3365	3.03	0.6565	3.01	0.06754	1.71	1.090	1.78
0.2219	0.07905	2.27	0.2053	1.82	0.03305	1.12	0.4636	1.34
0.1137	0.01990	2.06	0.1093	0.94	0.01654	1.04	0.2221	1.10
0.05754	0.004986	2.03	0.07361	0.58	0.008272	1.02	0.1102	1.03

Table 5.3
Absolute errors for the isotropic test problem with $\Delta t = h^2$.

h	$E_{\infty,2,u}$	eoc	$E_{\infty,2,\nu}$	eoc	$E_{2,2,w}$	eoc	$E_{2,\mathcal{E},\nabla w}$	eoc
1.	1.523	-	0.5929	-	0.1119	-	3.135	-
0.7368	0.5954	3.08	0.1827	3.85	0.4998	-4.90	2.203	1.16
0.4203	0.5108	0.27	0.06818	1.76	0.1906	1.72	0.9358	1.53
0.2219	0.1661	1.76	0.03228	1.17	0.06028	1.80	0.4549	1.13
0.1137	0.04476	1.96	0.01622	1.03	0.01591	1.99	0.2234	1.06
0.05754	0.01146	2.00	0.008113	1.02	0.004031	2.02	0.1112	1.02

Table 5.4
Absolute errors for the isotropic test problem with $\Delta t = h^2$.

h	$E_{\infty,\infty,u}$	eoc	$E_{2,2,V}$	eoc	$E_{\infty,2,\nabla u}$	eoc	$E_{2,2,\nabla w}$	eoc
1.	1.003	-	0.9711	-	0.5960	-	3.116	-
0.7368	0.3781	3.19	0.7349	0.91	0.1854	3.82	2.189	1.16
0.4203	0.2202	0.96	0.6887	0.12	0.06911	1.76	0.9296	1.53
0.2219	0.07354	1.72	0.2628	1.51	0.03292	1.16	0.4522	1.13
0.1137	0.01989	1.96	0.1163	1.22	0.01654	1.03	0.2221	1.06
0.05754	0.005087	2.00	0.05621	1.07	0.008271	1.02	0.1105	1.03

Table 5.5
Absolute errors for the anisotropic test problem with $\Delta t = h^2$.

h	$E_{\infty,2,u}$	eoc	$E_{\infty,2,\nu}$	eoc	$E_{2,2,w}$	eoc	$E_{2,\mathcal{E},\nabla w}$	eoc
0.1250	0.1475e-1	-	0.1354e-1	-	0.1409e-1	-	0.1207e-3	-
0.7138e-1	0.4999e-2	1.93	0.8346e-2	0.86	0.3483e-2	2.50	0.5734e-4	1.33
0.3807e-1	0.1458e-2	1.96	0.4399e-2	1.02	0.8862e-3	2.18	0.1997e-4	1.68
0.1964e-1	0.3937e-3	1.98	0.2216e-2	1.04	0.2221e-3	2.09	0.6971e-5	1.59
0.9969e-2	0.1032e-3	1.98	0.1110e-2	1.02	0.5553e-4	2.05	0.3079e-5	1.21

Table 5.6
Absolute errors for the anisotropic test problem with $\Delta t = h^2$.

h	$E_{\infty,\infty,u}$	eoc	$E_{2,2,V}$	eoc	$E_{\infty,2,\nabla u}$	eoc	$E_{2,2,\nabla w}$	eoc
0.1250	0.8037e-1	-	0.7644e-1	-	0.1547e-1	-	0.1200e-3	-
0.7138e-1	0.2658e-1	1.98	0.4285e-1	1.03	0.9390e-2	0.89	0.5710e-4	1.33
0.3807e-1	0.7753e-2	1.96	0.2293e-1	0.99	0.4894e-2	1.04	0.1988e-4	1.68
0.1964e-1	0.2093e-2	1.98	0.1182e-1	1.00	0.2453e-2	1.04	0.6921e-5	1.59
0.9969e-2	0.5481e-3	1.98	0.5997e-2	1.00	0.1227e-2	1.02	0.3080e-5	1.22

on the domain $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 0.125\}$ and for $t \in [0, 0.125]$. Domain and time interval have to be relatively small in order to remain in the setting of a graph. As in the isotropic case we have used a right-hand side g, and since u does not satisfy the natural boundary condition, we have extended the concept to include the inhomogeneous Neumann boundary condition $\langle \gamma'(\nabla u, -1), (\nu_{\partial\Omega}, 0)\rangle = c$ for a given

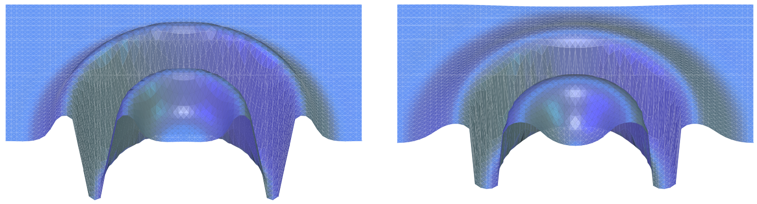

FIG. 5.1. *Initial function which leads to loss of the graph property after short time and solution becoming vertical (cut along the x_1-x_3 plane of symmetry).*

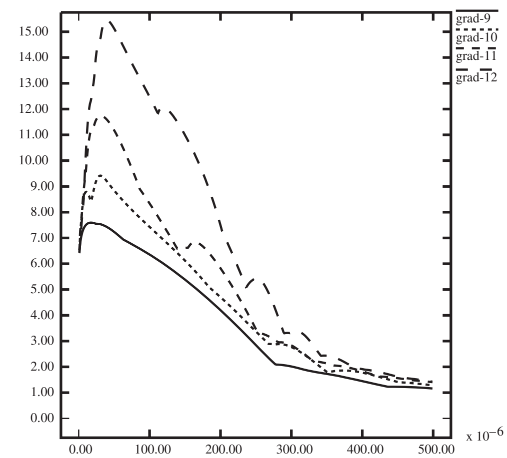

FIG. 5.2. *Lipschitz-norm of the discrete solution (vertical axis) plotted as a function of time* $t \in [0, 0.0005]$ *for the initial function from* (5.1) *for different spatial discretization levels.*

function c on $\partial\Omega$. As anisotropy we have used

$$\gamma(p) = \sqrt{0.25p_1^2 + p_2^2 + p_3^2}$$

and the stabilizing parameter was $\lambda = 1$.

We add an example of a surface which moves under isotropic surface diffusion and which loses its graph property in finite time. Nevertheless the discrete solution exists for all times. In Figure 5.1 two steps of the evolution are shown. In Figure 5.2 the maxima of the moduli of the gradients of the discrete solution is plotted as a function of time. The computational domain is $\Omega = (-1, 1)^2$ and the time interval is $[0, 0.0005]$. The graph of the solution becomes vertical after a short time, but the discrete solution continues to exist. We show the maximal gradient for the discretization levels 9, 10, 11, and 12. Observe that the number $1/h$ is 8.0, 11.32, 16.0, and 22.63 for these levels and by comparison with peaks in the graph of Figure 5.2 we see the suggestion of "infinite" gradients.

5.3. Numerical experiments. We end this section with two illustrative computations. First, we demonstrate the smoothing property of isotropic surface diffusion by choosing a highly oscillatory initial function u_0,

$$u_0(x) = 1 + 0.1\big(\sin(2(m+1)\pi x_1) + \sin(2m\pi x_1)(\sin(2(m+1)\pi x_2) + \sin(2m\pi x_2))\big) \tag{5.2}$$

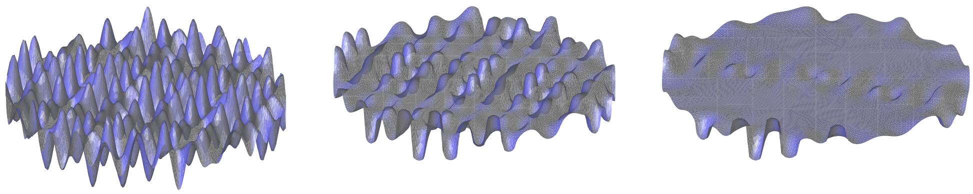

FIG. 5.3. *Solution u for the initial data* (5.2) *at times* 0.0, 3.5×10^{-6}, *and* 6.3×10^{-6}.

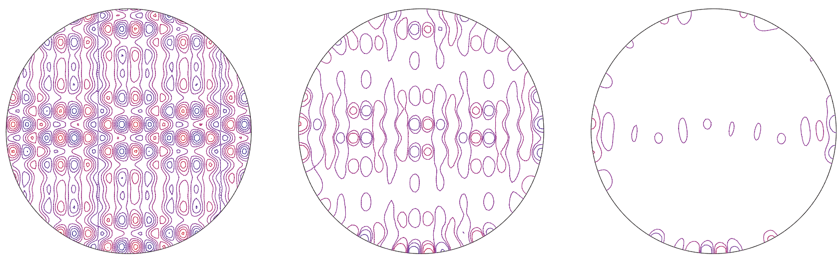

FIG. 5.4. *Level lines of the solution from Figure* 5.3.

with $m = 4$. The computational domain is the unit disk $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$, and we have used natural boundary conditions. The grid has to be fine in order to capture the frequency of the initial function. In order to show the rapid smoothing of u_0 we have chosen an extremely small time step proportional to h^4. In Figure 5.3 we show the solution at the times 0.0, 7.0×10^{-6} and 1.4×10^{-5}. Figure 5.4 shows level lines of the solution for these time steps. The level lines are equally distributed between the values 0.65 and 1.35 and are the same in all three cases.

Second, we computed an example for anisotropic surface diffusion with an extremely strong anisotropy. The anisotropy is chosen to be a regularized l^1 norm,

$$\gamma(p) = \sum_{j=1}^{3} \sqrt{p_j^2 + \varepsilon^2 |p|^2}, \tag{5.3}$$

where we have chosen $\varepsilon = 10^{-3}$. Thus the Frank diagram is a smoothed octahedron and the Wulff shape is a smoothed cube. The initial data were taken to depend on three random numbers $r_1, r_2, r_3 \in (0, 1)$,

$$\begin{aligned} u_0(x) = {} & \frac{1}{4}\left(\sin(2\pi r_1 x_1) + \frac{1}{4}\sin(3\pi r_2 x_2)\right)(0.1\sin(2\pi r_3 x_1) + \sin(5\pi r_1 x_2)) \\ & \times \sin(2\pi r_2 x_1 x_2). \end{aligned} \tag{5.4}$$

We used Neumann boundary conditions and the right-hand side (for the curvature equation) $f = 1 - x_1^2 - x_2^2$. The domain is given as $\Omega = (-1, 1) \times (-1, 1)$, and the triangulation contains 16641 vertices and 32768 triangles. We chose $\lambda = 4$. In Figure 5.5 we show the graph of the solution u in the direction of the x_1-axis. Figure 5.6 shows the graph for the time steps 0, 50, and 200. The Wulff shape (a smooth cube) appears in the solution as a consequence of the right-hand side f.

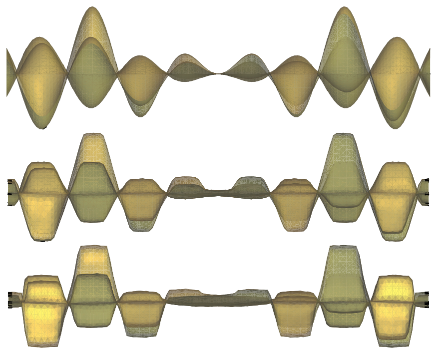

FIG. 5.5. *Anisotropic surface diffusion for the initial function* (5.4) *with anisotropy* (5.3), *viewed from the* x_1*-axis. Time steps* 0, 50, 200.

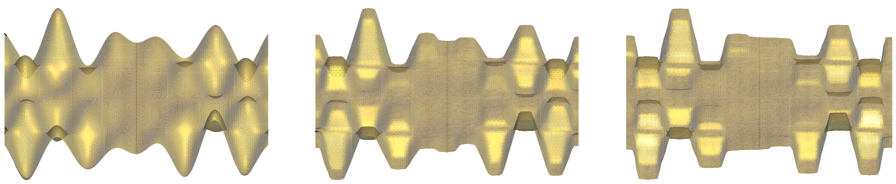

FIG. 5.6. *The solution from Figure* 5.5 *shown as graph.*

REFERENCES

[1] E. BÄNSCH, P. MORIN, AND R.H. NOCHETTO, *Surface diffusion of graphs: Variational formulation, error analysis, and simulation*, SIAM J. Numer. Anal., 42 (2004), pp. 773–799.

[2] G. BELLETTINI AND M. PAOLINI, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J., 25 (1996), pp. 537–566.

[3] A.J. BERNOFF, A.L. BERTOZZI, AND T.P. WITELSKI, *Axisymmetric surface diffusion: Dynamics and stability of self-similar pinchoff*, J. Statist. Phys., 93 (1998), pp. 725–776.

[4] J.W. CAHN, C.M. ELLIOTT, AND A. NOVICK-COHEN, *The Cahn-Hilliard equation with a concentration dependent mobility: Motion by minus the Laplacian of the mean curvature*, European J. Appl. Math., 7 (1996), pp. 287–301.

[5] J.W. CAHN AND J.E. TAYLOR, *Surface motion by surface diffusion*, Acta metall. mater., 42 (1994), pp. 1045–1063.

[6] W.C. CARTER, A.R. ROOSEN, J.W. CAHN, AND J.E. TAYLOR, *Shape evolution by surface diffusion and surface attachment limited kinetics on completely faceted surfaces*, Acta metall. mater., 43 (1995), pp. 4309–4323.

[7] B.D. COLEMAN, R.S. FALK, AND M. MOAKHER, *Stability of cylindrical bodies in the theory of surface diffusion*, Phys. D, 89 (1995), pp. 123–135.

[8] B.D. COLEMAN, R.S. FALK, AND M. MOAKHER, *Space-time finite element methods for surface diffusion with applications to the theory of the stability of cylinders*, SIAM J. Sci. Comput., 17 (1996), pp. 1434–1448.

[9] K. DECKELNICK AND G. DZIUK, *Discrete anisotropic curvature flow of graphs*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 1203–1222.

[10] K. DECKELNICK AND G. DZIUK, *Error estimates for a semi-implicit fully discrete finite element scheme for the mean curvature flow of graphs*, Interfaces Free Bound., 2 (2000), pp. 341–359.

[11] K. DECKELNICK AND G. DZIUK, *A fully discrete numerical scheme for weighted mean curvature flow*, Numer. Math., 91 (2002), pp. 423–452.

[12] K. DECKELNICK, G. DZIUK, AND C.M. ELLIOTT, *Error analysis of a semidiscrete numerical scheme for diffusion in axially symmetric surfaces*, SIAM J. Numer. Anal., 41 (2003), pp. 2161–2179.

[13] G. DZIUK, *Numerical schemes for the mean curvature flow of graphs*, in IUTAM Symposium on Variations of Domains and Free-Boundary Problems in Solid Mechanics, P. Argoul, M.

Frémond, and Q.S. Nguyen, eds., Kluwer Academic Publishers, Dordrecht-Boston-London, 1999, pp. 63–70.

[14] C.M. Elliott, D.A. French, and F.A. Milner, *A second order splitting method for the Cahn-Hilliard equation*, Numer. Math., 54 (1989), pp. 575–590.

[15] C.M. Elliott and H. Garcke, *Existence results for diffusive surface motion laws*, Adv. Math. Sci. Appl., 7 (1997), pp. 467–490.

[16] C.M. Elliott and S. Maier-Paape, *Losing a graph with surface diffusion*, Hokkaido Math. J., 30 (2001), pp. 297–305.

[17] J. Escher, U.F. Mayer, and G. Simonett, *The surface diffusion flow for immersed hypersurfaces*, SIAM J. Math. Anal., 29 (1998), pp. 1419–1433.

[18] Y. Giga and K. Ito, *On pinching of curves moved by surface diffusion*, Commun. Appl. Anal., 2 (1998), pp. 393–405.

[19] C. Herring, *Surface diffusion as a motivation for sintering*, in The Physics of Powder Metallurgy, W.E. Kingston, ed., McGraw Hill, New York, 1951, pp. 143–179.

[20] U.F. Mayer and G. Simonett, *Self-intersections for the surface diffusion and the volume-preserving mean curvature flow*, Differential Integral Equations, 13 (2000), pp. 1189–1199.

[21] W.W. Mullins, *Theory of thermal grooving*, J. Appl. Phys., 28 (1957), pp. 333–339.

[22] F.A. Nichols and W.W. Mullins, *Surface–(interface–) and volume–diffusion contributions to morphological changes driven by capillarity*, Trans. Metall. Soc. AIME, 233 (1965), pp. 1840–1847.

[23] P. Smereka, *Semi-implicit level set methods for curvature and surface diffusion motion*, J. Sci. Comput., 19 (2003), pp. 439–456.

[24] J.E. Taylor, J.W. Cahn, and C.A. Handwerker, *Geometric models of crystal growth*, Acta metall. mater., 40 (1992), pp. 1443–1474.

[25] H. Wong, M.J. Miksis, P.W. Voorhees, and S.H. Davis, *Universal pinch off of rods by capillarity-driven surface diffusion*, Scripta Mater., 39 (1998), pp. 55–60.