

ERROR ANALYSIS OF A SEMIDISCRETE NUMERICAL SCHEME FOR DIFFUSION IN AXIALLY SYMMETRIC SURFACES*

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Abstract. We analyze a semidiscrete numerical scheme for approximating the evolution of axially symmetric surfaces by surface diffusion. The fourth order equation is split into two coupled second order problems, which are approximated by linear finite elements. We prove error bounds for the resulting scheme and present numerical test calculations that confirm our analysis.

Key words. surface diffusion, finite elements, error estimates, fourth order parabolic equation

AMS subject classifications. 65N30, 35K55

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1. Introduction. In recent years motion by mean curvature has been extensively studied from the computational point of view. However, the related curvature flow of motion by the surface Laplacian has received far less attention in the numerical analysis literature. The geometrical problem is to find a time-dependent surface $\Gamma(t)$ evolving according to the law of motion

$$(1.1) \quad V = \Delta_{\Gamma(t)}\kappa \quad \text{on } \Gamma(t),$$

where V and κ denote, respectively, the normal velocity and the mean curvature of the surface. Our sign convention is that κ with respect to the outer normal is positive for spheres. The Laplace–Beltrami or surface Laplacian operator for Γ is denoted by Δ_{Γ} . This evolution has interesting geometrical properties: if $\Gamma(t)$ is a closed surface bounding a domain $\Omega(t)$, then the volume of $\Omega(t)$ is preserved and the surface area of $\Gamma(t)$ decreases. It is known that for closed curves in the plane or closed surfaces in \mathbb{R}^3 balls are asymptotically stable subject to small perturbations; see [9], [10]. However, it is also known that topological changes such as pinch-off are possible [11], [13].

Equation (1.1) is referred to as a surface diffusion equation because it models the diffusion of mass within the bounding surface of a solid body. At the atomistic level atoms on the surface move along the surface due to a driving force consisting of a chemical potential difference. For a surface with constant surface energy density the appropriate chemical potential in this setting is the mean curvature κ . This leads to the flux law

$$\rho V = -\operatorname{div}_{\Gamma}\mathbf{j},$$

where ρ is the mass density and \mathbf{j} is the mass flux in the surface, with the constitutive flux law [12], [14]

$$\mathbf{j} = -D\nabla_{\Gamma}\kappa.$$

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Here, D is the diffusion constant. From these equations we obtain the law (1.1) after an appropriate nondimensionalization. The notion of surface diffusion is due to Mullins [14] and for a review we refer to [2].

In applications one is interested in the stability of so-called whiskers, which are axially symmetric cylindrical bodies of small diameter with respect to their length; see [15], [3], [1], and [16]. We shall be concerned with an axially symmetric cylindrical body, whose boundary

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = (x, r(x, t) \cos \phi, r(x, t) \sin \phi), x \in [0, L], \phi \in [0, 2\pi]\}$$

evolves by surface diffusion. We assume that the radius r is a smooth positive function, which is periodic in x , so that $r(0, t) = r(L, t)$. In these coordinates the mean curvature of $\Gamma(t)$ is

$$(1.2) \quad \kappa = \frac{1}{r\sqrt{1+r_x^2}} - \frac{r_{xx}}{\sqrt{1+r_x^2}^3} = \frac{1}{r\sqrt{1+r_x^2}} - \left(\frac{r_x}{\sqrt{1+r_x^2}} \right)_x,$$

while the normal velocity and surface Laplacian of the mean curvature of the surface, respectively, are given by

$$V = \frac{r_t}{\sqrt{1+r_x^2}}, \quad \Delta_\Gamma \kappa = \frac{1}{r\sqrt{1+r_x^2}} \left(\frac{r\kappa_x}{\sqrt{1+r_x^2}} \right)_x.$$

It follows from these two equations that r satisfies the quasi-linear fourth order parabolic problem

$$(1.3) \quad r_t = \frac{1}{r} \left(\frac{r\kappa_x}{\sqrt{1+r_x^2}} \right)_x \quad \text{in } I \times (0, T],$$

$$(1.4) \quad r(0, t) = r(L, t) \quad \text{in } (0, T],$$

$$(1.5) \quad \kappa(0, t) = \kappa(L, t) \quad \text{in } (0, T],$$

$$(1.6) \quad r(\cdot, 0) = r_0 \quad \text{in } I,$$

where $I = (0, L)$ and κ is given by (1.2). The initial function r_0 is assumed to be periodic and positive.

Our concern in this paper is the analysis of a finite element discretization based on the above natural splitting of the fourth order problem into two coupled second order equations for the radial variable r and the mean curvature κ . We note that [4] proposed a similar second order splitting scheme and used $R = r^2$ and κ as the variables. Our principal result is an error estimate for the spatial discretization, which is actually attained in numerical experiments.

The paper is organized as follows: in section 2 we introduce the numerical scheme, prove the local existence and uniqueness of the discrete solution, and formulate our main error estimate. This result is proved in section 3, while section 4 contains numerical tests.

2. The discrete problem. As already mentioned in the introduction, our discretization of (1.3) is based on the idea of splitting the elliptic part, which is of fourth order, into two second order operators. This is similar in spirit to the second order splitting techniques proposed for the numerical approximation of the Cahn–Hilliard

equation in [8]. To begin, we deduce from (1.2)

$$(2.1) \quad r\kappa = \frac{1}{\sqrt{1+r_x^2}} - r \left(\frac{r_x}{\sqrt{1+r_x^2}} \right)_x = \sqrt{1+r_x^2} - \left(\frac{rr_x}{\sqrt{1+r_x^2}} \right)_x.$$

Thus (1.3) and (2.1) allow the variational formulation

$$(2.2) \quad \int_I rr_t \eta dx = - \int_I \frac{r\kappa_x \eta_x}{\sqrt{1+r_x^2}} dx \quad \forall \eta \in H^1_{per}(I),$$

$$(2.3) \quad \int_I r\kappa \zeta dx = \int_I \sqrt{1+r_x^2} \zeta dx + \int_I \frac{rr_x \zeta_x}{\sqrt{1+r_x^2}} dx \quad \forall \zeta \in H^1_{per}(I),$$

where $H^1_{per}(I) = \{\eta \in H^1(I) \mid \eta(0) = \eta(L)\}$. We employ (2.2), (2.3) in order to define a semidiscrete scheme using linear finite elements to approximate r and κ . Let $0 = x_0 < x_1 < \dots < x_N = L$, $h_j := x_j - x_{j-1}$, and $h := \max_{1 \leq j \leq N} h_j$. We shall make an inverse assumption of the form

$$(2.4) \quad h \leq \rho h_j \quad \forall j = 1, \dots, N,$$

where $\rho > 0$ is independent of h . The space of linear finite elements is defined by

$$X_h := \{\phi_h \in C^0(\bar{I}) \mid \phi_h|_{[x_{j-1}, x_j]} \in P^1, 1 \leq j \leq N, \phi_h(0) = \phi_h(L)\}.$$

Our discrete problem now reads as follows: find $r_h, \kappa_h : [0, T] \rightarrow X_h$ such that

$$(2.5) \quad \int_I r_h r_{h,t} \eta_h dx = - \int_I \frac{r_h \kappa_{h,x} \eta_{h,x}}{\sqrt{1+r_{h,x}^2}} dx \quad \forall \eta_h \in X_h, t \in (0, T],$$

$$(2.6) \quad \int_I r_h \kappa_h \zeta_h dx = \int_I \sqrt{1+r_{h,x}^2} \zeta_h dx + \int_I \frac{r_h r_{h,x} \zeta_{h,x}}{\sqrt{1+r_{h,x}^2}} dx \quad \forall \zeta_h \in X_h, t \in [0, T],$$

$$(2.7) \quad r_h(0) = I_h r_0,$$

where I_h denotes the Lagrange interpolation operator.

LEMMA 2.1. *There exists $T_h > 0$ such that (2.5)–(2.7) has a unique solution $(r_h, \kappa_h) \in C^1([0, T_h]; X_h \times X_h)$ satisfying $\frac{1}{2} \min_{[0, L]} r_0 \leq r_h \leq 2 \max_{[0, L]} r_0$ in $[0, L] \times [0, T_h]$.*

Proof. Choose a smooth globally Lipschitz-continuous function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ with the properties $\beta(s) = s$ for $\frac{1}{2} \min_{[0, L]} r_0 \leq s \leq 2 \max_{[0, L]} r_0$, $\frac{1}{4} \min_{[0, L]} r_0 \leq \beta(s) \leq 4 \max_{[0, L]} r_0$ for all $s \in \mathbb{R}$. We first consider the following modified problem: find $r_h, \kappa_h : [0, T] \rightarrow X_h$ such that

$$(2.8) \quad \int_I \beta(r_h) r_{h,t} \eta_h dx = - \int_I \frac{r_h \kappa_{h,x} \eta_{h,x}}{\sqrt{1+r_{h,x}^2}} dx \quad \forall \eta_h \in X_h, t \in [0, T],$$

$$(2.9) \quad \int_I \beta(r_h) \kappa_h \zeta_h dx = \int_I \sqrt{1+r_{h,x}^2} \zeta_h dx + \int_I \frac{r_h r_{h,x} \zeta_{h,x}}{\sqrt{1+r_{h,x}^2}} dx \quad \forall \zeta_h \in X_h, t \in [0, T],$$

$$(2.10) \quad r_h(0) = I_h r_0.$$

Denoting by ψ_1, \dots, ψ_N the usual nodal basis of X_h , we can represent (r_h, κ_h) as

$$(2.11) \quad r_h(\cdot, t) = \sum_{j=1}^N r_j(t) \psi_j, \quad \kappa_h(\cdot, t) = \sum_{j=1}^N \kappa_j(t) \psi_j$$

and write $\underline{r}(t) = (r_1(t), \dots, r_N(t))^T$, $\underline{\kappa}(t) = (\kappa_1(t), \dots, \kappa_N(t))^T$. In view of the properties of β we may rewrite (2.9) in the form $\underline{\kappa}(t) = G(\underline{r}(t))$ with a Lipschitz-continuous mapping $G : \mathbb{R}^N \rightarrow \mathbb{R}^N$. Inserting this into (2.8) and using again the properties of β , we may write this relation as

$$\underline{r}'(t) = F(\underline{r}(t)), \quad \underline{r}(0) = (r_0(x_1), \dots, r_0(x_N))^T,$$

with a Lipschitz-continuous $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$. The existence and uniqueness of \underline{r} on some interval $[0, T_h]$ follows directly from the theory of ODEs. The corresponding functions r_h and κ_h given by (2.11) will then solve (2.8)–(2.10). Since $r_h(0) = I_h r_0$ and by making T_h smaller if necessary, we may assume that $\frac{1}{2} \min_{[0,L]} r_0 \leq r_h \leq 2 \max_{[0,L]} r_0$ in $[0, L] \times [0, T_h]$ so that, in view of the properties of β , (r_h, κ_h) also solves (2.5)–(2.7). \square

Using $\eta_h = \kappa_h$ in (2.5) and $\zeta_h = r_{h,t}$ in (2.6) and taking the difference of the resulting equations, we obtain

$$\begin{aligned} 0 &= \int_I \sqrt{1 + r_{h,x}^2} r_{h,t} dx + \int_I \frac{r_h r_{h,x} r_{h,tx}}{\sqrt{1 + r_{h,x}^2}} dx + \int_I \frac{r_h \kappa_{h,x}^2}{\sqrt{1 + r_{h,x}^2}} dx \\ &= \frac{d}{dt} \int_I r_h \sqrt{1 + r_{h,x}^2} dx + \int_I \frac{r_h \kappa_{h,x}^2}{\sqrt{1 + r_{h,x}^2}} dx. \end{aligned}$$

Thus

$$(2.12) \quad \sup_{0 \leq t \leq T_h} \int_I r_h \sqrt{1 + r_{h,x}^2} dx + \int_0^{T_h} \int_I \frac{r_h \kappa_{h,x}^2}{\sqrt{1 + r_{h,x}^2}} dx dt \leq C(r_0).$$

Before we formulate an error estimate for the scheme (2.5)–(2.7), we state a local existence and uniqueness result for the continuous problem.

THEOREM 2.2. *Suppose that $r_0 \in H^4_{per}(I)$ is strictly positive. Then there exists $T_0 > 0$ such that (1.3)–(1.6) has a unique solution (r, κ) , which satisfies $r \in L^\infty(0, T_0; H^4_{per}(I))$, $r_t \in L^2(0, T_0; H^2_{per}(I))$, and $r(x, t) > 0$ for all $(x, t) \in I \times [0, T_0]$.*

Proof. A similar result was proved in [11] for a formulation of (1.1) in terms of the distance function to a fixed reference curve. Since the resulting equation has the same structure as (1.3)–(1.6), the methods employed in [11] can be applied to our situation. \square

We denote by $[0, T_{\max})$, $T_{\max} \in (0, \infty]$ the maximal time interval on which the solution from Theorem 2.2 exists and fix $T < T_{\max}$. Then there exist constants $0 < c_0 \leq C_0$ and $M \geq 0$ (depending on T) such that

$$(2.13) \quad c_0 \leq r \leq C_0, \quad |r_x| \leq C_0 \quad \text{on } [0, L] \times [0, T],$$

$$(2.14) \quad \sup_{t \in (0, T)} \|r(\cdot, t)\|_{H^4(I)}^2 + \int_0^T \|r_t\|_{H^2(I)}^2 dt \leq M^2.$$

Combining these bounds with (1.2), (1.3), and the inequality $\|f\|_{L^\infty(I)} \leq C\|f\|_{H^1(I)}$,

we note for later use

$$(2.15) \quad \|\kappa(\cdot, t)\|_{H^{1,\infty}(I)} + \|\kappa(\cdot, t)\|_{H^2(I)} + \|r_t(\cdot, t)\|_{L^2(I)} \leq C \quad \text{uniformly in } t \in [0, T],$$

where C depends on L, c_0, C_0 , and M .

Our main result is the following error estimate, the proof of which will be given in the next section.

THEOREM 2.3. *There exists an $h_0 > 0$ such that for all $0 < h \leq h_0$ the discrete solution (r_h, κ_h) exists on $[0, T]$ and*

$$(2.16) \quad \sup_{0 \leq t \leq T} \|(r - r_h)(t)\|_{H^1(I)}^2 + \int_0^T \|\kappa - \kappa_h\|_{H^1(I)}^2 dt \leq Ch^2.$$

The constant C depends on L, T, c_0, C_0, M , and ρ .

3. Proof of Theorem 2.3. Let us define

$$\hat{T}_h := \sup \left\{ t \in [0, T] \mid (r_h, \kappa_h) \text{ solves (2.5)–(2.7) on } [0, t] \text{ and } \frac{1}{2}c_0 \leq r_h \leq 2C_0, |r_{h,x}| \leq 2C_0 \text{ on } [0, t] \right\}.$$

By choosing T_h smaller if necessary (in order to satisfy the bound on $r_{h,x}$), we may deduce from Lemma 2.1 that $\hat{T}_h > 0$. Our aim is to show that $\hat{T}_h = T$ for small h . This will be achieved by proving the bounds (2.16) on $[0, \hat{T}_h]$, which will enable us to continue the discrete solution. By the definition of \hat{T}_h we have

$$(3.1) \quad \frac{1}{2}c_0 \leq r_h \leq 2C_0, \quad |r_{h,x}| \leq 2C_0 \text{ on } [0, L] \times [0, \hat{T}_h].$$

In what follows, we shall denote by C a constant which may depend on L, T, c_0, C_0, M , and ρ . Additional dependencies of C will be stated explicitly. We start with a useful auxiliary lemma.

LEMMA 3.1. *Let $v \in H^1_{per}(I), t \in [0, \hat{T}_h]$. Then we have for $\epsilon > 0$*

$$\left| \int_I \frac{r_h}{r} v r r_t dx - \int_I v r_h r_{h,t} dx \right| \leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|v\|_{H^1(I)}^2 + Ch^2 + C \|r - r_h\|_{H^1(I)}^2.$$

Proof. Fix $t \in [0, \hat{T}_h]$ and denote by $Q_h : L^2(I) \rightarrow X_h$ the following weighted projection: for a given $u \in L^2(I)$ let $Q_h u \in X_h$ be defined by

$$(3.2) \quad \int_I r_h u \zeta_h dx = \int_I r_h Q_h u \zeta_h dx \quad \forall \zeta_h \in X_h.$$

We claim that

$$(3.3) \quad \|u - Q_h u\|_{L^2(I)} + h \|u_x - (Q_h u)_x\|_{L^2(I)} \leq Ch \|u_x\|_{L^2(I)} \quad \forall u \in H^1_{per}(I).$$

To see this, we first note that (3.1), (3.2), and an interpolation inequality imply

$$\begin{aligned} \frac{c_0}{2} \int_I |u - Q_h u|^2 &\leq \int_I r_h (u - Q_h u)(u - Q_h u) = \int_I r_h (u - Q_h u)(u - I_h u) \\ &\leq 2C_0 \|u - Q_h u\|_{L^2(I)} h \|u_x\|_{L^2(I)}, \end{aligned}$$

which yields the first part of (3.3). In view of (2.4) we have that $\|v_{h,x}\|_{L^2(I)} \leq Ch^{-1}\|v_h\|_{L^2(I)}$ for $v_h \in X_h$, and therefore

$$\begin{aligned} \|u_x - (Q_h u)_x\|_{L^2(I)} &\leq \|u_x - (I_h u)_x\|_{L^2(I)} + \|(I_h u)_x - (Q_h u)_x\|_{L^2(I)} \\ &\leq 2\|u_x\|_{L^2(I)} + Ch^{-1}\|I_h u - Q_h u\|_{L^2(I)} \\ &\leq 2\|u_x\|_{L^2(I)} + Ch^{-1}(\|u - I_h u\|_{L^2(I)} + \|u - Q_h u\|_{L^2(I)}) \\ &\leq C\|u_x\|_{L^2(I)}, \end{aligned}$$

where we used the bound on $\|u - Q_h u\|_{L^2(I)}$. This proves (3.3).

Next we infer from (3.2) and (2.5) that

$$\int_I v r_h r_{h,t} dx = \int_I Q_h v r_h r_{h,t} dx = - \int_I \frac{r_h \kappa_{h,x} (Q_h v)_x}{\sqrt{1+r_{h,x}^2}} dx \quad \forall v \in H_{per}^1(I).$$

If we combine this relation with (2.2), we may continue with

$$\begin{aligned} &\int_I \frac{r_h}{r} v r r_t dx - \int_I v r_h r_{h,t} dx \\ &= - \int_I \frac{r \kappa_x (\frac{r_h}{r} v)_x}{\sqrt{1+r_x^2}} dx + \int_I \frac{r_h \kappa_{h,x} (Q_h v)_x}{\sqrt{1+r_{h,x}^2}} dx \\ &= - \int_I \frac{r \kappa_x v}{\sqrt{1+r_x^2}} \frac{r_{h,x} r - r_x r_h}{r^2} dx + \int_I \frac{r - r_h}{\sqrt{1+r_x^2}} \kappa_x v_x dx \\ &\quad + \int_I \left(\frac{r_h}{\sqrt{1+r_{h,x}^2}} - \frac{r}{\sqrt{1+r_x^2}} \right) \kappa_{h,x} (Q_h v)_x dx \\ &\quad + \int_I \frac{r}{\sqrt{1+r_x^2}} (\kappa_{h,x} - \kappa_x) (Q_h v)_x dx + \int_I \frac{r \kappa_x}{\sqrt{1+r_x^2}} (Q_h v - v)_x dx \\ &\equiv \sum_{i=1}^5 S_i. \end{aligned}$$

In view of (2.13), (2.15), and (3.1), we then have

$$\begin{aligned} |S_1| &\leq C \int_I |v| (|r - r_h| + |r_x - r_{h,x}|) dx \leq \|v\|_{L^2(I)}^2 + C \|r - r_h\|_{H^1(I)}^2, \\ |S_2| &\leq \|v_x\|_{L^2(I)}^2 + C \|r - r_h\|_{L^2(I)}^2. \end{aligned}$$

Next, (2.15) and (3.3) imply

$$\begin{aligned} |S_3| &\leq \int_I \left| \frac{r_h}{\sqrt{1+r_{h,x}^2}} - \frac{r}{\sqrt{1+r_x^2}} \right| (|\kappa_x| + |\kappa_{h,x} - \kappa_x|) |(Q_h v)_x| dx \\ &\leq C \int_I (|r - r_h| + |r_x - r_{h,x}|) |(Q_h v)_x| dx + C \int_I |\kappa_x - \kappa_{h,x}| |(Q_h v)_x| dx \\ &\leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|v_x\|_{L^2(I)}^2 + C \|r - r_h\|_{H^1(I)}^2, \end{aligned}$$

and similarly,

$$|S_4| \leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|v_x\|_{L^2(I)}^2.$$

Finally, integration by parts, (1.3), (2.15), and (3.3) yield

$$\begin{aligned} |S_5| &= \left| - \int_I \left(\frac{r \kappa_x}{\sqrt{1+r_x^2}} \right)_x (Q_h v - v) dx \right| \\ &\leq Ch \|v_x\|_{L^2(I)} \|r_t\|_{L^2(I)} \leq Ch^2 + C \|v_x\|_{L^2(I)}^2. \end{aligned}$$

Collecting the above estimates concludes the proof of the lemma. \square

As a first application of the above result we derive a differential inequality for the L^2 -error.

LEMMA 3.2.

$$\frac{1}{2} \frac{d}{dt} \|r - r_h\|_{L^2(I)}^2 \leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|r - r_h\|_{H^1(I)}^2 + Ch^2.$$

Proof. Clearly,

$$\begin{aligned} (3.4) \quad \frac{1}{2} \frac{d}{dt} \|r - r_h\|_{L^2(I)}^2 &= \int_I (r - r_h)(r_t - r_{h,t}) dx \\ &= \int_I \frac{1}{r} (r - r_h) r r_t dx - \int_I \frac{1}{r_h} (r - r_h) r_h r_{h,t} dx. \end{aligned}$$

If we apply Lemma 3.1 to the function

$$v := \frac{1}{r_h} (r - r_h)(\cdot, t) \quad \text{for } t \in (0, \hat{T}_h),$$

the result follows. \square

The main part of the proof of Theorem 2.3 consists in controlling the H^1 -seminorms of $r - r_h$ and $\kappa - \kappa_h$. The idea is to mimic the argument which led to the a priori estimate (2.12) in such a way that it can be applied to the difference between exact and discrete solution. This suggests using $\eta_h = I_h \kappa - \kappa_h$, $\zeta_h = I_h r_t - r_{h,t}$ in the error relations satisfied by $r - r_h$, $\kappa - \kappa_h$. In order to derive these relations we use $\eta = \eta_h \in X_h$ in (2.2) and $\zeta = \zeta_h \in X_h$ in (2.3) and take the difference with (2.5), (2.6), respectively. This leads to

$$(3.5) \quad \int_I (r r_t - r_h r_{h,t}) \eta_h dx = - \int_I \left(\frac{r \kappa_x}{\sqrt{1+r_x^2}} - \frac{r_h \kappa_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) \eta_{h,x} dx \quad \forall \eta_h \in X_h,$$

$$\begin{aligned} (3.6) \quad \int_I (r \kappa - r_h \kappa_h) \zeta_h dx &= \int_I (\sqrt{1+r_x^2} - \sqrt{1+r_{h,x}^2}) \zeta_h dx \\ &\quad + \int_I \left(\frac{r r_x}{\sqrt{1+r_x^2}} - \frac{r_h r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) \zeta_{h,x} dx \quad \forall \zeta_h \in X_h. \end{aligned}$$

LEMMA 3.3. *We have for all $\epsilon > 0$*

$$\begin{aligned} (3.7) \quad \frac{d}{dt} \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x} r_x + 1}{\sqrt{1+r_x^2}} \right) dx &+ \int_I \frac{r_h}{\sqrt{1+r_{h,x}^2}} (\kappa_x - \kappa_{h,x})^2 dx \\ &\leq C_\epsilon \|\kappa - \kappa_h\|_{H^1(I)}^2 + C_\epsilon (1 + \|r_t\|_{H^2(I)}) \|r - r_h\|_{H^1(I)}^2 + C_\epsilon h^2 (1 + \|r_t\|_{H^2(I)}^2). \end{aligned}$$

Proof. Using $\zeta_h = I_h r_t - r_{h,t}$ in (3.6), we obtain

$$\begin{aligned}
 (3.8) \quad \int_I (r\kappa - r_h\kappa_h)(I_h r_t - r_{h,t})dx &= \int_I (\sqrt{1+r_x^2} - \sqrt{1+r_{h,x}^2})(I_h r_t - r_{h,t})dx \\
 &+ \int_I r_h \left(\frac{r_x}{\sqrt{1+r_x^2}} - \frac{r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) (r_{tx} - r_{h,tx})dx \\
 &+ \int_I (r - r_h) \frac{r_x}{\sqrt{1+r_x^2}} (r_{tx} - r_{h,tx})dx \\
 &+ \int_I \left(\frac{rr_x}{\sqrt{1+r_x^2}} - \frac{r_h r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) ((I_h r_t)_x - r_{tx})dx.
 \end{aligned}$$

Note first that the second integral can be written as

$$\begin{aligned}
 &\int_I r_h \left(\frac{r_x}{\sqrt{1+r_x^2}} - \frac{r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) (r_{tx} - r_{h,tx})dx \\
 &= \int_I r_h \frac{\partial}{\partial t} \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) dx \\
 &+ \int_I r_h r_{t,x} \left(\frac{r_{h,x}}{\sqrt{1+r_x^2}} - \frac{r_{h,x}}{\sqrt{1+r_{h,x}^2}} + \frac{r_x}{\sqrt{1+r_x^2}} - \frac{1+r_x r_{h,x}}{1+r_x^2} \frac{r_x}{\sqrt{1+r_x^2}} \right) dx \\
 &= \frac{d}{dt} \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) dx - \int_I r_{h,t} \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) dx \\
 &+ \int_I r_h r_{t,x} \left(\frac{r_{h,x}}{\sqrt{1+r_x^2}} - \frac{r_{h,x}}{\sqrt{1+r_{h,x}^2}} + \frac{r_x}{\sqrt{1+r_x^2}} - \frac{1+r_x r_{h,x}}{1+r_x^2} \frac{r_x}{\sqrt{1+r_x^2}} \right) dx.
 \end{aligned}$$

Integration by parts together with (1.2) implies for the third term in (3.8)

$$\begin{aligned}
 &\int_I (r - r_h) \frac{r_x}{\sqrt{1+r_x^2}} (r_{tx} - r_{h,tx})dx \\
 &= - \int_I (r_x - r_{h,x}) \frac{r_x}{\sqrt{1+r_x^2}} (r_t - r_{h,t})dx - \int_I (r - r_h) \left(\frac{r_x}{\sqrt{1+r_x^2}} \right)_x (r_t - r_{h,t})dx \\
 &= - \int_I r_t (r_x - r_{h,x}) \frac{r_x}{\sqrt{1+r_x^2}} dx + \int_I r_{h,t} \sqrt{1+r_x^2} dx - \int_I r_{h,t} \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} dx \\
 &- \int_I (r - r_h) \frac{1}{r\sqrt{1+r_x^2}} (r_t - r_{h,t})dx + \int_I (r - r_h) \kappa (r_t - r_{h,t})dx.
 \end{aligned}$$

Inserting the above equations into (3.8), we derive

$$\begin{aligned}
 (3.9) \quad \int_I (r\kappa - r_h\kappa_h)(I_h r_t - r_{h,t})dx &= \frac{d}{dt} \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) dx \\
 &+ \int_I (\sqrt{1+r_x^2} - \sqrt{1+r_{h,x}^2}) I_h r_t dx - \int_I r_t (r_x - r_{h,x}) \frac{r_x}{\sqrt{1+r_x^2}} dx
 \end{aligned}$$

$$\begin{aligned}
 & + \int_I r_h r_{tx} \left(\frac{r_{h,x}}{\sqrt{1+r_x^2}} - \frac{r_{h,x}}{\sqrt{1+r_{h,x}^2}} + \frac{r_x}{\sqrt{1+r_x^2}} - \frac{1+r_x r_{h,x}}{1+r_x^2} \frac{r_x}{\sqrt{1+r_x^2}} \right) dx \\
 & - \int_I (r-r_h) \frac{1}{r\sqrt{1+r_x^2}} (r_t-r_{h,t}) dx + \int_I (r-r_h) \kappa(r_t-r_{h,t}) dx \\
 & + \int_I \left(\frac{r r_x}{\sqrt{1+r_x^2}} - \frac{r_h r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) ((I_h r_t)_x - r_{tx}) dx.
 \end{aligned}$$

Let us next insert $\eta_h = I_h \kappa - \kappa_h$ into (3.5):

$$\begin{aligned}
 (3.10) \quad & \int_I (r r_t - r_h r_{h,t}) (I_h \kappa - \kappa_h) dx \\
 & = - \int_I \left(\frac{r \kappa_x}{\sqrt{1+r_x^2}} - \frac{r_h \kappa_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) ((I_h \kappa)_x - \kappa_{h,x}) dx \\
 & = \int_I \left(\frac{r \kappa_x}{\sqrt{1+r_x^2}} - \frac{r_h \kappa_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) (\kappa_x - (I_h \kappa)_x) dx \\
 & \quad - \int_I \frac{r_h}{\sqrt{1+r_{h,x}^2}} (\kappa_x - \kappa_{h,x})^2 dx \\
 & \quad - \int_I \left(\frac{r}{\sqrt{1+r_x^2}} - \frac{r_h}{\sqrt{1+r_{h,x}^2}} \right) \kappa_x (\kappa_x - \kappa_{h,x}) dx.
 \end{aligned}$$

Combining (3.9) and (3.10), we obtain

$$(3.11) \quad \frac{d}{dt} \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x} r_x + 1}{\sqrt{1+r_x^2}} \right) dx + \int_I \frac{r_h}{\sqrt{1+r_{h,x}^2}} (\kappa_x - \kappa_{h,x})^2 dx = \sum_{i=1}^8 \tilde{S}_i,$$

where

$$\begin{aligned}
 \tilde{S}_1 &= \int_I (r \kappa - r_h \kappa_h) (I_h r_t - r_{h,t}) dx - \int_I (r r_t - r_h r_{h,t}) (I_h \kappa - \kappa_h) dx \\
 & \quad - \int_I (r - r_h) \kappa (r_t - r_{h,t}) dx, \\
 \tilde{S}_2 &= - \int_I r_t \left(\sqrt{1+r_x^2} - \sqrt{1+r_{h,x}^2} - (r_x - r_{h,x}) \frac{r_x}{\sqrt{1+r_x^2}} \right) dx, \\
 \tilde{S}_3 &= - \int_I r_h r_{tx} \left(\frac{r_{h,x}}{\sqrt{1+r_x^2}} - \frac{r_{h,x}}{\sqrt{1+r_{h,x}^2}} + \frac{r_x}{\sqrt{1+r_x^2}} - \frac{1+r_x r_{h,x}}{1+r_x^2} \frac{r_x}{\sqrt{1+r_x^2}} \right) dx, \\
 \tilde{S}_4 &= \int_I (r - r_h) \frac{1}{r\sqrt{1+r_x^2}} (r_t - r_{h,t}) dx, \\
 \tilde{S}_5 &= \int_I (\sqrt{1+r_x^2} - \sqrt{1+r_{h,x}^2}) (r_t - I_h r_t) dx,
 \end{aligned}$$

$$\begin{aligned} \tilde{S}_6 &= - \int_I \left(\frac{rr_x}{\sqrt{1+r_x^2}} - \frac{r_h r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) ((I_h r_t)_x - r_{tx}) dx, \\ \tilde{S}_7 &= \int_I \left(\frac{r\kappa_x}{\sqrt{1+r_x^2}} - \frac{r_h \kappa_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) (\kappa_x - (I_h \kappa)_x) dx, \\ \tilde{S}_8 &= - \int_I \left(\frac{r}{\sqrt{1+r_x^2}} - \frac{r_h}{\sqrt{1+r_{h,x}^2}} \right) \kappa_x (\kappa_x - \kappa_{h,x}) dx. \end{aligned}$$

The terms $\tilde{S}_1, \dots, \tilde{S}_8$ have been organized in such a way that each of them is quadratic in an appropriate difference. To see this, let us examine them in more detail. First,

$$\begin{aligned} \tilde{S}_1 &= \int_I (r\kappa - r_h \kappa_h)(r_t - r_{h,t}) dx - \int_I (rr_t - r_h r_{h,t})(\kappa - \kappa_h) dx - \int_I (r - r_h)\kappa(r_t - r_{h,t}) dx \\ &\quad + \int_I (r\kappa - r_h \kappa_h)(I_h r_t - r_t) dx - \int_I (rr_t - r_h r_{h,t})(I_h \kappa - \kappa) dx \\ &= - \int_I r_t(\kappa - \kappa_h)(r - r_h) dx + \int_I (r\kappa - r_h \kappa_h)(I_h r_t - r_t) dx - \int_I (rr_t - r_h r_{h,t})(I_h \kappa - \kappa) dx \\ &\equiv A_1 + A_2 + A_3. \end{aligned}$$

Using an interpolation estimate, (2.15), and the continuous embedding $H^1(I) \hookrightarrow L^\infty(I)$, we obtain

$$\begin{aligned} |A_1 + A_2| &\leq C \|r_t\|_{L^2(I)} \|\kappa - \kappa_h\|_{L^2(I)} \|r - r_h\|_{L^\infty(I)} + Ch \|r_{tx}\|_{L^2(I)} \|r\kappa - r_h \kappa_h\|_{L^2(I)} \\ &\leq \epsilon \|\kappa - \kappa_h\|_{L^2(I)}^2 + C_\epsilon \|r - r_h\|_{H^1(I)}^2 + C_\epsilon h^2 \|r_{tx}\|_{L^2(I)}^2, \end{aligned}$$

while

$$A_3 = \int_I (I_h \kappa - \kappa) r_h r_{h,t} dx - \int_I \frac{r_h}{r} (I_h \kappa - \kappa) r r_t dx + \int_I (I_h \kappa - \kappa) \left(\frac{r_h}{r} - 1 \right) r r_t dx.$$

We infer from Lemma 3.1 with $v = \kappa - I_h \kappa$ and well-known interpolation estimates that

$$\begin{aligned} |A_3| &\leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|\kappa - I_h \kappa\|_{H^1(I)}^2 + Ch^2 + C \|r - r_h\|_{H^1(I)}^2 \\ &\quad + C \|r_t\|_{L^2(I)} \|r - r_h\|_{L^\infty(I)} \|\kappa - I_h \kappa\|_{L^2(I)} \\ &\leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon h^2 \|\kappa\|_{H^2(I)}^2 + C \|r - r_h\|_{H^1(I)}^2. \end{aligned}$$

Recalling (2.15), we conclude

$$|\tilde{S}_1| \leq \epsilon \|\kappa - \kappa_h\|_{H^1(I)}^2 + C_\epsilon \|r - r_h\|_{H^1(I)}^2 + C_\epsilon (1 + \|r_{tx}\|_{L^2(I)}^2) h^2.$$

Next, observing that

$$(3.12) \quad \left| \sqrt{1+q^2} - \sqrt{1+p^2} - (q-p) \frac{q}{\sqrt{1+q^2}} \right| \leq C(q-p)^2 \quad \forall q, p \in \mathbb{R},$$

we obtain

$$|\tilde{S}_2| \leq C \|r_t\|_{L^\infty(I)} \|r_x - r_{h,x}\|_{L^2(I)}^2 \leq C \|r_t\|_{H^1(I)} \|r_x - r_{h,x}\|_{L^2(I)}^2.$$

Let us now examine \tilde{S}_3 . A short calculation shows

$$\begin{aligned} & \frac{p}{\sqrt{1+q^2}} - \frac{p}{\sqrt{1+p^2}} + \frac{q}{\sqrt{1+q^2}} - \frac{1+pq}{1+q^2} \frac{q}{\sqrt{1+q^2}} \\ &= \frac{p(1+q^2)(\sqrt{1+p^2} - \sqrt{1+q^2}) - q^2\sqrt{1+p^2}(p-q)}{\sqrt{1+q^2}^3\sqrt{1+p^2}} \\ &= \frac{p}{\sqrt{1+q^2}\sqrt{1+p^2}} \left(\sqrt{1+p^2} - \sqrt{1+q^2} - (p-q)\frac{p}{\sqrt{1+p^2}} \right) \\ & \quad + \frac{p-q}{\sqrt{1+q^2}^3(1+p^2)}(p^2(1+q^2) - q^2(1+p^2)), \end{aligned}$$

which implies in view of (3.12)

$$\left| \frac{p}{\sqrt{1+q^2}} - \frac{p}{\sqrt{1+p^2}} + \frac{q}{\sqrt{1+q^2}} - \frac{1+pq}{1+q^2} \frac{q}{\sqrt{1+q^2}} \right| \leq C(p-q)^2$$

for all $p, q \in \mathbb{R}$. Therefore,

$$|\tilde{S}_3| \leq C\|r_{tx}\|_{L^\infty(I)}\|r_x - r_{h,x}\|_{L^2(I)}^2 \leq C\|r_{tx}\|_{H^1(I)}\|r_x - r_{h,x}\|_{L^2(I)}^2.$$

If we write

$$\tilde{S}_4 = \int_I (r - r_h) \frac{1}{r\sqrt{1+r_x^2}} (r_t - r_{h,t}) dx = \int_I \frac{r_h}{r} v r r_t dx - \int_I v r_h r_{h,t} dx$$

with $v = \frac{r-r_h}{r_h r \sqrt{1+r_x^2}}$ and apply Lemma 3.1, we deduce

$$\begin{aligned} |\tilde{S}_4| &\leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \left\| \frac{r - r_h}{r_h r \sqrt{1+r_x^2}} \right\|_{H^1(I)}^2 + Ch^2 + C\|r - r_h\|_{H^1(I)}^2 \\ &\leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|r - r_h\|_{H^1(I)}^2 + Ch^2. \end{aligned}$$

In view of interpolation estimates, Young's inequality, and (2.15),

$$|\tilde{S}_5| \leq Ch\|r_{tx}\|_{L^2(I)}\|r_x - r_{h,x}\|_{L^2(I)} \leq Ch^2\|r_{tx}\|_{L^2(I)}^2 + C\|r_x - r_{h,x}\|_{L^2(I)}^2,$$

$$|\tilde{S}_6| \leq Ch\|r_{txx}\|_{L^2(I)}\|r - r_h\|_{H^1(I)} \leq Ch^2\|r_{txx}\|_{L^2(I)}^2 + C\|r - r_h\|_{H^1(I)}^2,$$

$$\begin{aligned} |\tilde{S}_7| &\leq Ch\|\kappa_{xx}\|_{L^2(I)}(\|\kappa_x - \kappa_{h,x}\|_{L^2(I)} + \|r - r_h\|_{H^1(I)}) \\ &\leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon h^2 + C\|r - r_h\|_{H^1(I)}^2. \end{aligned}$$

Finally,

$$|\tilde{S}_8| \leq \epsilon \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon \|r - r_h\|_{H^1(I)}^2.$$

If we insert the above estimates for $\tilde{S}_1, \dots, \tilde{S}_8$ into (3.11), the result is

$$\begin{aligned} & \frac{d}{dt} \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) dx + \int_I \frac{r_h}{\sqrt{1+r_{h,x}^2}} (\kappa_x - \kappa_{h,x})^2 dx \\ & \leq C_\epsilon \|\kappa - \kappa_h\|_{H^1(I)}^2 + C_\epsilon (1 + \|r_t\|_{H^2(I)}) \|r - r_h\|_{H^1(I)}^2 + C_\epsilon (1 + \|r_t\|_{H^2(I)}) h^2, \end{aligned}$$

which completes the proof of the lemma. \square

Remark 3.4. (a) In order to interpret the integral

$$(3.13) \quad \int_I r_h \left(\sqrt{1 + r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1 + r_x^2}} \right) dx$$

occurring in (3.7), we note that

$$\nu = \frac{1}{\sqrt{1 + r_x^2}} (-r_x, \cos \phi, \sin \phi), \quad \nu_h = \frac{1}{\sqrt{1 + r_{h,x}^2}} (-r_{h,x}, \cos \phi, \sin \phi)$$

are the unit outward normals to

$$\Gamma(t) = \{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = (x, r(x, t) \cos \phi, r(x, t) \sin \phi), x \in [0, L], \phi \in [0, 2\pi] \},$$

$$\Gamma_h(t) = \{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = (x, r_h(x, t) \cos \phi, r_h(x, t) \sin \phi), x \in [0, L], \phi \in [0, 2\pi] \},$$

respectively. Observing that $dS = r_h \sqrt{1 + r_{h,x}^2} dx d\phi$ is the surface element on Γ_h , a short calculation shows that

$$\int_I r_h \left(\sqrt{1 + r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1 + r_x^2}} \right) dx = \frac{1}{2\pi} \int_{\Gamma_h} |\nu - \nu_h|^2 dS.$$

A similar relation was used in [5], [6] in an error analysis for the mean curvature flow of graphs.

(b) Under the conditions (2.13) and (3.1), the expression (3.13) is equivalent to $\|r_x - r_{h,x}\|_{H^1(I)}^2$. To see this, note that

$$\begin{aligned} & \sqrt{1 + r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1 + r_x^2}} \\ &= \frac{(\sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2} - (r_{h,x}r_x + 1))(\sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2} + (r_{h,x}r_x + 1))}{\sqrt{1 + r_x^2}(\sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2} + (r_{h,x}r_x + 1))} \\ &= \frac{(r_x - r_{h,x})^2}{\sqrt{1 + r_x^2}(\sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2} + (r_{h,x}r_x + 1))}, \end{aligned}$$

which implies

$$(3.14) \quad \frac{c_0}{4(1 + C_0^2)\sqrt{1 + 4C_0^2}} \|r_x - r_{h,x}\|_{H^1(I)}^2 \leq \int_I r_h \left(\sqrt{1 + r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1 + r_x^2}} \right) dx \leq C_0 \|r_x - r_{h,x}\|_{H^1(I)}^2,$$

since

$$\begin{aligned} 1 & \leq \sqrt{1 + r_x^2} (\sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2} + (r_{h,x}r_x + 1)) \\ & \leq \sqrt{1 + r_x^2} (\sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2} + \sqrt{1 + r_{h,x}^2} \sqrt{1 + r_x^2}) \leq 2(1 + C_0^2)\sqrt{1 + 4C_0^2}. \end{aligned}$$

It remains to derive an estimate for $\|\kappa - \kappa_h\|_{L^2(I)}$.

LEMMA 3.5.

$$\|\kappa - \kappa_h\|_{L^2(I)} \leq C(\|r - r_h\|_{H^1(I)} + \|\kappa_x - \kappa_{h,x}\|_{L^2(I)} + h).$$

Proof. Clearly,

$$\begin{aligned} & \int_I r_h(\kappa - \kappa_h)^2 dx \\ &= -\int_I (r - r_h)\kappa(\kappa - \kappa_h) dx + \int_I (r\kappa - r_h\kappa_h)(\kappa - I_h\kappa) dx + \int_I (r\kappa - r_h\kappa_h)(I_h\kappa - \kappa_h) dx. \end{aligned}$$

Using (3.6) in order to rewrite the third integral, we deduce

$$\begin{aligned} & \int_I r_h(\kappa - \kappa_h)^2 dx = -\int_I (r - r_h)\kappa(\kappa - \kappa_h) dx + \int_I (r\kappa - r_h\kappa_h)(\kappa - I_h\kappa) dx \\ & \quad + \int_I (I_h\kappa - \kappa_h)(\sqrt{1+r_x^2} - \sqrt{1+r_{h,x}^2}) dx + \int_I \left(\frac{rr_x}{\sqrt{1+r_x^2}} - \frac{r_h r_{h,x}}{\sqrt{1+r_{h,x}^2}} \right) (I_h\kappa - \kappa_h)_x dx \\ & \leq C\|r - r_h\|_{L^2(I)}\|\kappa - \kappa_h\|_{L^2(I)} + C(\|r - r_h\|_{L^2(I)} + \|\kappa - \kappa_h\|_{L^2(I)})\|\kappa - I_h\kappa\|_{L^2(I)} \\ & \quad + C\|I_h\kappa - \kappa_h\|_{L^2(I)}\|r_x - r_{h,x}\|_{L^2(I)} + C\|(I_h\kappa)_x - \kappa_{h,x}\|_{L^2(I)}\|r - r_h\|_{H^1(I)} \\ & \leq \epsilon\|\kappa - \kappa_h\|_{L^2(I)}^2 + C_\epsilon\|r - r_h\|_{H^1(I)}^2 + C_\epsilon h^2 + C\|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2. \end{aligned}$$

Here we have again used (2.15). Choosing $\epsilon = \frac{c_0}{4}$ and recalling (3.1), we complete the proof of the lemma. \square

We are now in position to complete the proof of Theorem 2.3. Combining Lemmas 3.2, 3.3, and 3.5 and (3.1), we obtain with $\lambda = \frac{c_0}{2\sqrt{1+4C_0^2}}$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|r - r_h\|_{L^2(I)}^2 + \frac{d}{dt} \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) dx + \lambda\|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 \\ & \leq C_\epsilon\|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 + C_\epsilon(1 + \|r_t\|_{H^2(I)})\|r - r_h\|_{H^1(I)}^2 + C_\epsilon(1 + \|r_t\|_{H^2(I)})h^2. \end{aligned}$$

Choosing ϵ sufficiently small and recalling (3.14), the function

$$\phi(t) := \frac{1}{2} \|(r - r_h)(t)\|_{L^2(I)}^2 + \int_I r_h \left(\sqrt{1+r_{h,x}^2} - \frac{r_{h,x}r_x + 1}{\sqrt{1+r_x^2}} \right) (t) dx$$

satisfies

$$\begin{aligned} (3.15) \quad & \phi'(t) + \frac{\lambda}{2} \|\kappa_x - \kappa_{h,x}\|_{L^2(I)}^2 \leq C(1 + \|r_t\|_{H^2(I)})h^2 + C(1 + \|r_t\|_{H^2(I)})\phi(t), \\ & 0 \leq t \leq \hat{T}_h. \end{aligned}$$

Now, (2.7) and (3.14) yield $\phi(0) \leq Ch^2$, so that Gronwall's lemma implies

$$\phi(t) \leq Ch^2 \left(1 + \int_0^T \|r_t\|_{H^2(I)}^2 dt \right) \exp \left(\int_0^T C(1 + \|r_t\|_{H^2(I)}) dt \right), \quad 0 \leq t \leq \hat{T}_h.$$

Therefore,

$$(3.16) \quad \sup_{0 < t < \hat{T}_h} \|(r - r_h)(t)\|_{H^1(I)} \leq Ch,$$

and, using (3.15) together with Lemma 3.5,

$$(3.17) \quad \int_0^{\hat{T}_h} \|\kappa - \kappa_h\|_{H^1(I)}^2 dt \leq Ch^2.$$

We can now prove that $\hat{T}_h = T$. If not, we would have $\hat{T}_h < T$; the smoothness of r , (3.16), and an inverse estimate then would imply that

$$\|(r - r_h)(t)\|_{H^1,\infty(I)} \leq C\sqrt{h}, \quad 0 \leq t \leq \hat{T}_h,$$

which combined with (2.13) would give

$$\frac{3}{4}c_0 \leq r_h \leq \frac{3}{2}C_0, \quad |r_{h,x}| \leq \frac{3}{2}C_0 \quad \text{in } I \times [0, \hat{T}_h]$$

provided that $h \leq h_0$ and h_0 is sufficiently small. However, then we could extend the discrete solution to an interval $[0, \hat{T}_h + \delta]$ for some $\delta > 0$ with

$$\frac{1}{2}c_0 \leq r_h \leq 2C_0, \quad |r_{h,x}| \leq 2C_0 \quad \text{in } I \times [0, \hat{T}_h + \delta],$$

which contradicts the definition of \hat{T}_h . Thus $\hat{T}_h = T$ for $h \leq h_0$ and (3.16), (3.17) imply our result.

4. Numerical results. We use the notation

$$\begin{aligned} r_j(t) &= r_h(x_j, t), \quad \kappa_j(t) = \kappa_h(x_j, t), \quad j = 0, \dots, N, \\ q_j(t) &= \sqrt{h_j^2 + (r_j(t) - r_{j-1}(t))^2}, \quad j = 1, \dots, N. \end{aligned}$$

The spatially discrete problem (2.5), (2.6) then is translated into the following system of ODEs. By a dot we denote the time derivative. For numerical tests we shall use an additional right-hand side f which we include in the equations here.

$$\begin{aligned} &\frac{h_j}{6}(r_{j-1} + r_j)\dot{r}_{j-1} + \left(\frac{h_j}{6}r_{j-1} + \frac{1}{2}(h_j + h_{j+1})r_j + \frac{h_{j+1}}{6}r_{j+1} \right)\dot{r}_j \\ &\quad + \frac{h_{j+1}}{6}(r_j + r_{j+1})\dot{r}_{j+1} - \frac{r_{j-1} + r_j}{q_j}\kappa_{j-1} + \left(\frac{r_{j-1} + r_j}{q_j} + \frac{r_j + r_{j+1}}{q_{j+1}} \right)\kappa_j \\ &\quad - \frac{r_j + r_{j+1}}{q_{j+1}}\kappa_{j+1} = \frac{1}{2}(q_j(r_{j-1} + r_j)f_{j-\frac{1}{2}} + q_{j+1}(r_j + r_{j+1})f_{j+\frac{1}{2}}), \\ &\frac{h_j}{6}(r_{j-1} + r_j)\kappa_{j-1} + \left(\frac{h_j}{6}r_{j-1} + \frac{1}{2}(h_j + h_{j+1})r_j + \frac{h_{j+1}}{6}r_{j+1} \right)\kappa_j \\ &\quad + \frac{h_{j+1}}{6}(r_j + r_{j+1})\kappa_{j+1} + \frac{r_{j-1} + r_j}{q_j}r_{j-1} - \left(\frac{r_{j-1} + r_j}{q_j} + \frac{r_j + r_{j+1}}{q_{j+1}} \right)r_j \\ &\quad + \frac{r_j + r_{j+1}}{q_{j+1}}r_{j+1} = q_j + q_{j+1} \end{aligned} \tag{4.1}$$

for $j = 1, \dots, N, t \in (0, T]$, with periodic boundary conditions and initial condition $r_j(0) = r_0(x_j), j = 0, \dots, N$. For the right-hand side term involving f we have used a simple integration formula and the notation $f_{j\pm\frac{1}{2}} = f((x_j + x_{j\pm 1})/2)$.

The time discretization is done via a semi-implicit scheme which also linearizes the problem. Furthermore we use mass lumping at suitable positions. Let $\tau > 0$ be the time step size and $M = \lceil T/\tau \rceil$. For a generic function w we denote by w^m ($0 \leq m \leq M$) the evaluation on the m th time level: $w^m = w(\cdot, m\tau)$. The fully discrete scheme then reads as follows.

Algorithm 4.1. Let $r_j^0 = r_0(x_j)$, $j = 0, \dots, N$. For $m = 1, \dots, M$ solve

$$\begin{aligned} & \frac{1}{\tau}(h_j + h_{j+1})r_j^{m-1}(r_j^m - r_j^{m-1}) \\ & - \frac{r_j^{m-1} + r_j^m}{q_j^{m-1}}\kappa_{j-1}^m + \left(\frac{r_{j-1}^{m-1} + r_j^{m-1}}{q_j^{m-1}} + \frac{r_j^{m-1} + r_{j+1}^{m-1}}{q_{j+1}^{m-1}} \right) \kappa_j^m - \frac{r_j^{m-1} + r_{j+1}^{m-1}}{q_{j+1}^{m-1}}\kappa_{j+1}^m \\ & = \frac{1}{2}(q_j^{m-1}(r_{j-1}^{m-1} + r_j^{m-1})f_{j-\frac{1}{2}}^m + q_{j+1}^{m-1}(r_j^{m-1} + r_{j+1}^{m-1})f_{j+\frac{1}{2}}^m), \\ & (h_j + h_{j+1})r_j^{m-1}\kappa_j^m \\ & + \frac{r_{j-1}^{m-1} + r_j^{m-1}}{q_j^{m-1}}r_{j-1}^m - \left(\frac{r_{j-1}^{m-1} + r_j^{m-1}}{q_j^{m-1}} + \frac{r_j^{m-1} + r_{j+1}^{m-1}}{q_{j+1}^{m-1}} \right) r_j^m + \frac{r_j^{m-1} + r_{j+1}^{m-1}}{q_{j+1}^{m-1}}r_{j+1}^m \\ & = q_j^{m-1} + q_{j+1}^{m-1} \end{aligned}$$

for $j = 1, \dots, N$, $m = 1, \dots, M$.

In every time step a linear system for $\underline{r}^m = (r_1^m, \dots, r_N^m)$ and $\underline{\kappa}^m = (\kappa_1^m, \dots, \kappa_N^m)$ of the form

$$(4.2) \quad \frac{1}{\tau}M^{m-1}\underline{r}^m + S^{m-1}\underline{\kappa}^m = \underline{c}^{m-1},$$

$$(4.3) \quad M^{m-1}\underline{\kappa}^m - S^{m-1}\underline{r}^m = \underline{d}^{m-1}$$

has to be solved. Here M^{m-1} is a suitable mass matrix, S^{m-1} is a stiffness matrix, and \underline{c}^{m-1} , \underline{d}^{m-1} are right-hand sides depending on the quantities of the $(m-1)$ st time step with built-in periodic boundary conditions. Note that the time discretization is semi-implicit with respect to the position r but is fully implicit with respect to curvature κ . The linear system (4.2), (4.3) was solved by inserting the second equation into the first one, which leads to the following linear system for \underline{r}^m :

$$(4.4) \quad \left(\frac{1}{\tau}M^{m-1} + S^{m-1}(M^{m-1})^{-1}S^{m-1} \right) \underline{r}^m = \underline{c}^{m-1} - S^{m-1}(M^{m-1})^{-1}\underline{d}^{m-1}.$$

Note that the matrix M^{m-1} is a diagonal matrix. The system (4.4) was solved by a conjugate gradient method.

For all computations we have used uniform spatial grids $h_j = h$ with h as indicated.

We test the scheme with a known continuous solution. We choose

$$r(x, t) = (1 + 0.25 \sin \pi(x - 1))(1 + 0.125 \cos t)$$

on the interval $I = [0, 2]$ for $T = 1$ and calculate the corresponding right-hand side f from (1.3) and (1.2). Now we are able to compute the error between continuous solution r , κ and discrete solution r_h^m , κ_h^m and calculate the experimental order of convergence from the errors for two grids. As time step size we have chosen $\tau = 0.1h^2$.

TABLE 4.1

Absolute errors in various norms and experimental orders of convergence (in brackets) for the test problem for the choice $\tau = 0.1h^2$.

N	h	$\ r - r_h\ _{L^\infty(H^1)}$	$\ \kappa - \kappa_h\ _{L^2(H^1)}$
20	0.1	0.3010	2.2669
40	0.05	0.1544 (0.96)	1.1693 (0.96)
80	0.025	0.07784 (0.99)	0.5892 (0.99)
160	0.0125	0.03903 (1.00)	0.2952 (1.00)
320	0.00625	0.01953 (1.00)	0.1477 (1.00)

TABLE 4.2

Absolute errors in various norms and experimental orders of convergence (in brackets) for the test problem for the choice $\tau = 0.1h$.

N	h	$\ r - r_h\ _{L^\infty(H^1)}$	$\ \kappa - \kappa_h\ _{L^2(H^1)}$
20	0.1	0.2575	2.2597
40	0.05	0.1399 (0.88)	1.1672 (0.95)
80	0.025	0.07363 (0.93)	0.5886 (0.99)
160	0.0125	0.03790 (0.96)	0.2950 (1.00)
320	0.00625	0.01922 (0.98)	0.1476 (1.00)

The results are shown in Table 4.1. We measured the errors

$$\|r - r_h\|_{L^\infty((0,T),H^1(I))} \quad \text{and} \quad \|\kappa - \kappa_h\|_{L^2((0,T),H^1(I))}.$$

The results confirm the error estimates in Theorem 2.3 precisely. A quite astonishing result is that these convergence results experimentally also hold in the case of linear coupling of time step size and spatial grid size (see Table 4.2), in particular, that no stability problems arise even though the scheme is only semi-implicit. This is in some sense similar to the case of mean curvature flow, for which in [7] stability of a semi-implicit scheme was proved without any time step restriction.

In [3] it was shown that solutions of axially symmetric surface diffusion may exhibit the following dynamical behavior: After an initial rapid decay, some perturbations slowly grow in amplitude and finally lead to pinch-off. We recomputed an example from [3], for which the initial surface is given by

$$(4.5) \quad r_0(x) = 1 + 0.05 \left(\sin \left(\frac{m+1}{2} x \right) + \sin \left(\frac{m}{2} x \right) \right), \quad x \in (0, n\pi).$$

Figure 4.1 shows the rapid decay of perturbations for $m = 10$. For better visibility we scaled the graphics vertically by 100.

For $m = 14$ we show the long time behavior of the solution $r = r(x, t)$. In order to make the dynamical behavior more transparent we plot the solution in Figure 4.2 for $t \in [0, 10]$ and in Figure 4.3 for $t \in [20, 27.861]$. We have used 400 nodes and a time step size $\tau = 0.1h^2$. Note that our error analysis is only valid as long as r is bounded away from zero. For calculations near the pinch-off singularity we adapted the time step according to $\tau = 0.1h^2 \min_{[0,4\pi]} r_h^3$, a criterion which was found experimentally. Finally, we computed the solution of axisymmetric surface diffusion for the initial surface given by

$$(4.6) \quad r_0(x) = 1 - 0.95 |x| \sin \frac{\pi}{x}, \quad x \in (-1, 1).$$

Here we have used 500 spatial nodes and a time discretization as in the previous example. The results are shown in Figure 4.4.

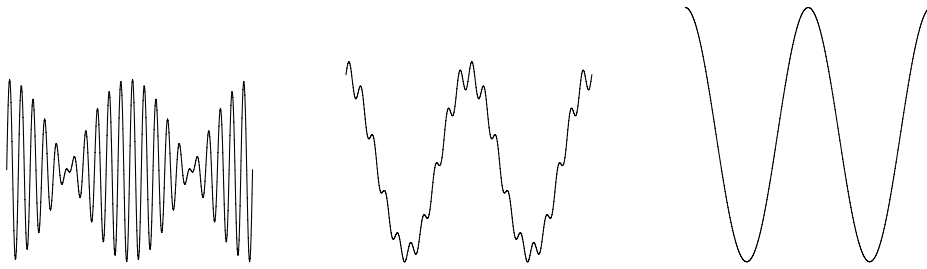


FIG. 4.1. Evolution of the initial surface given by (4.5) with $m = 10$, $n = 8$ for $t = 0.0, 0.01, 0.1$, vertically scaled by 100.

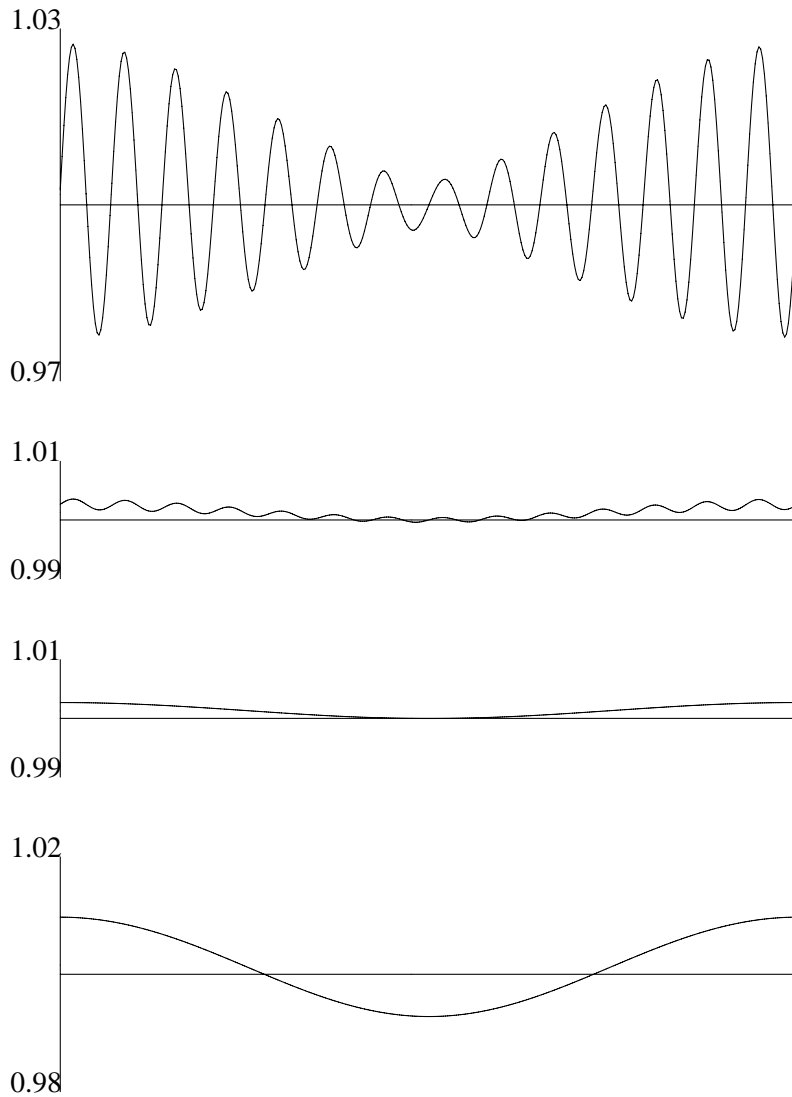


FIG. 4.2. Evolution of the axially symmetric initial surface given by (4.5) with $m = 14$, $n = 4$ under surface diffusion. The horizontal axis runs from 0 to 4π , and the vertical axis is scaled by 100. Time steps $t = 0.00, 0.0014, 0.10, \text{ and } 10.0$.

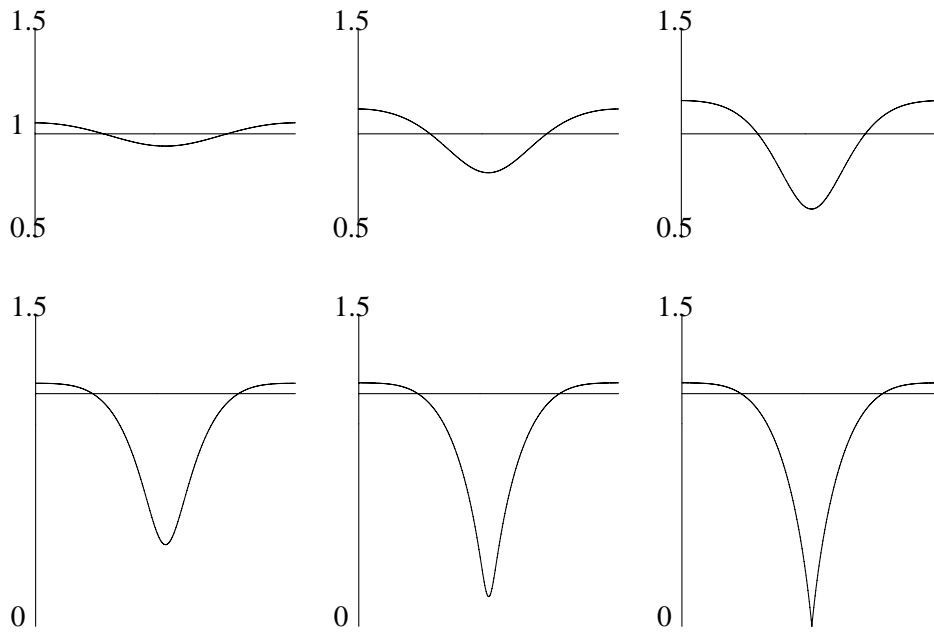


FIG. 4.3. Evolution of the axially symmetric initial surface given by (4.5) with $m = 14$, $n = 4$ under surface diffusion. The horizontal axis runs from 0 to 4π , and the vertical axis is scaled by 10. Time steps $t = 20.0, 25.0, 27.0, 27.75, 27.86$, and 27.861 .

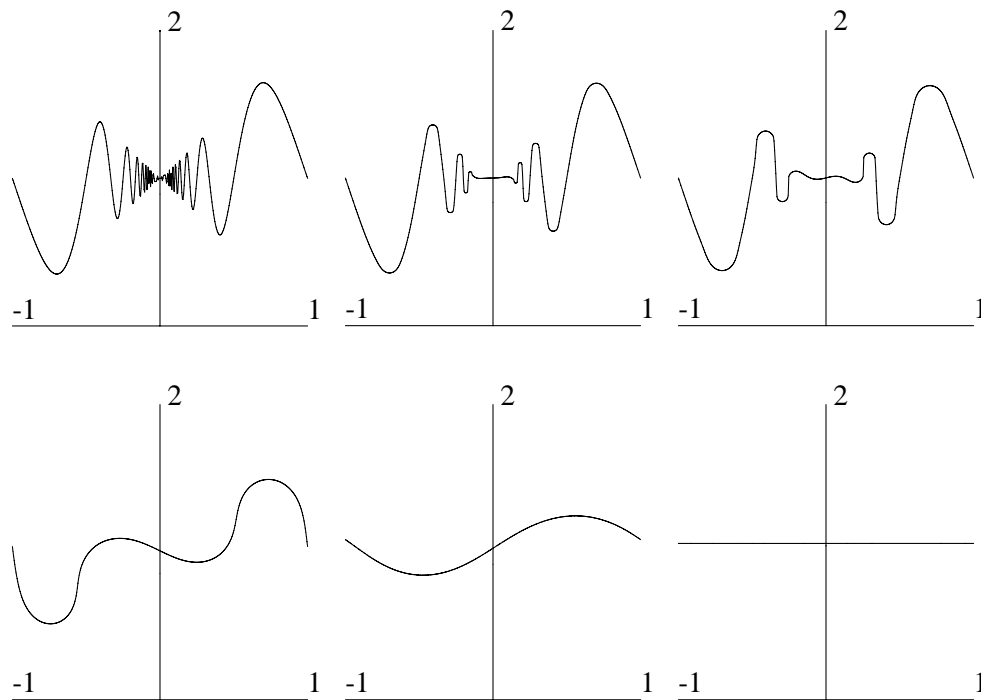


FIG. 4.4. Evolution of the axially symmetric initial surface given by (4.6) under surface diffusion. Time steps $t = 0.00, 6.26 \cdot 10^{-7}, 7.59 \cdot 10^{-6}, 6.97 \cdot 10^{-4}, 6.45 \cdot 10^{-3}$, and $9.82 \cdot 10^{-2}$.

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