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A FINITE ELEMENT APPROXIMATION OF A VARIATIONAL INEQUALITY FORMULATION OF BEAN'S MODEL FOR SUPERCONDUCTIVITY*

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Abstract. We introduce a finite element approximation of a variational formulation of Bean's model for the physical configuration of an infinitely long cylindrical superconductor subject to a transverse magnetic field. We prove an error between the exact solution and the approximate solution for the current density and the magnetic field in appropriate norms of order $h^{1/2} + \Delta t$. Numerical simulations for a variety of applied magnetic fields are also presented.

Key words. finite elements, variational inequalities, superconductors

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1. Introduction.

In this paper we consider the numerical approximation of an evolutionary variational inequality arising from a critical state model for a type-II superconductor. The physical setting is that of an infinitely long cylinder of type-II superconducting material subject to an applied transverse magnetic field. We take the cylindrical superconductor to occupy the region $D = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$, a bounded, simply connected domain in $\mathbb{R}^2$, is the cross section of the superconductor. The physical vector fields that are relevant are the current density $\mathbf{J} = (0, 0, J(\underline{x}, t))$, which is parallel to the axis of the cylinder, and the magnetic field $\mathbf{H} = (\underline{H}(\underline{x}, t), 0)$, which is orthogonal to the cylinder's axis, for $\underline{x} \in \mathbb{R}^2$. The well-known Bean critical state model can be formulated as an evolutionary variational inequality for $J(\underline{x}, t)$ of the form (see [10]):

(**P**) Find $J(\cdot, t) \in K$ for a.e. $t \geq 0$ such that $J(\cdot, 0) = J_0 \in K$ and

$$\left(\frac{\partial GJ}{\partial t}, \eta - J\right) \geq (f, \eta - J) \quad \forall\ \eta \in K. \tag{1.1}$$

Here $(\cdot, \cdot)$ denotes the standard L^2 inner product over Ω,

$$\mathcal{V} := \Big\{\eta \in L^2_{\rm loc}(\mathbb{R}^2) : \nabla\eta \in L^2(\mathbb{R}^2),\ (\eta, 1) = 0\Big\},$$
$$K = \Big\{\eta \in \mathcal{V} : \eta = 0 \ \text{ on } \mathbb{R}^2/\overline{\Omega}, |\eta| \leq J_c, \ \ (\eta, 1) = 0\Big\}$$

and $G : \mathcal{V}' \to \mathcal{V}$ is the "inverse Laplacian" operator defined by the solution to the following variational problem:

Given $v \in \mathcal{V}'$, find $Gv \in \mathcal{V}$ such that

$$(\nabla Gv, \nabla\eta)_{\mathbb{R}^2} = \langle v, \eta\rangle \quad \forall\ \eta \in \mathcal{V} \tag{1.2}$$

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with $\langle\cdot,\cdot\rangle$ denoting the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$. For $v \in \mathcal{F} \subset \mathcal{V}'$ we have

$$\langle v, \eta\rangle = (v, \eta) \quad \forall\, \eta \in \mathcal{V},$$

where

$$\mathcal{F} := \Big\{\eta \in \mathcal{V}' : \eta \in L^2_{\text{loc}}(\mathbb{R}^2) :\ \eta = 0 \text{ on } \mathbb{R}^2/\overline{\Omega}\Big\}.$$

Setting

$$\mathcal{F}_0 := \Big\{\eta \in \mathcal{F} : (\eta, 1) = 0\Big\},$$

we have the following for all $v \in \mathcal{F}_0$:

$$-\Delta G v = v \quad \text{in } \mathbb{R}^2, \qquad \int_\Omega G v \, d\underline{x} = 0, \qquad \text{and} \qquad \nabla G v \sim 0 \quad \text{at } \infty, \tag{1.3}$$

and Gv is unique.

Throughout the remaining sections we assume that

$$f \in L^2(0,T; H^2(\Omega)), \quad f_t \in L^2(0,T; H^1(\Omega)). \tag{1.4}$$

It follows from the classical theory of evolutionary variational inequalities that (**P**) has a unique solution; see [10, 5].

2. Derivation of the model and reduction to a bounded domain.

2.1. Derivation of the model. We suppose that all field variables depend only on t and $\underline{x} \in \mathbb{R}^2$, and that there is a prescribed, time dependent, smooth magnetic field $\mathbf{H}^a = (\underline{H}^a(\underline{x}, t), 0)$ applied at infinity and a prescribed, bounded current density $\mathbf{J}^a = J^a(\underline{x}, t)\mathbf{e}_3$, exterior to the superconductor, such that the compatibility condition

$$\mathbf{J}^a - \text{curl } \mathbf{H}^a \to 0 \quad \text{as } |\underline{x}| \to \infty$$

is satisfied. Then Maxwell's equations, neglecting displacement current, are

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial t} + \text{curl } \mathbf{E} = \mathbf{0} & \qquad \text{in } \mathbb{R}^2, \\ \text{curl } \mathbf{H} = \mathbf{J} & \qquad \text{in } \mathbb{R}^2, \\ \nabla \cdot \mathbf{H} = 0 & \qquad \text{in } \mathbb{R}^2, \end{aligned}$$

where $\mathbf{E}$ is the electric field; see [10]. Note we have taken the magnetic permeability equal to 1 for simplicity.

The critical state model assumes the following nonlinear Ohm's law in the superconductor,

$$\mathbf{E} \;=\; \rho \mathbf{J} \qquad \text{in } \Omega$$

with

$$|\mathbf{J}| \;\leq\; J_c \qquad \text{in } \Omega,$$

and the effective resistivity ρ achieves the constraint on $|\mathbf{J}|$ by the relation $\rho \in \beta(|\mathbf{J}|)$, where β is a multivalued map given by the graph

$$\beta(r) = \begin{cases} (-\infty, 0] & \text{if} \quad r = -J_c, \\ 0 & \text{if} \quad |r| < J_c, \\ [0, \infty) & \text{if} \quad r = J_c. \end{cases}$$

We assume that exterior to the superconductor the current is prescribed so

$$\mathbf{J} = \mathbf{J}^a \qquad \text{in } \mathbb{R}^2/\overline{\Omega}.$$

To complete this set of equations we require initial and boundary conditions for the magnetic field given, respectively, by

$$\mathbf{H}(\underline{x}, 0) \;=\; \mathbf{H}_0(\underline{x})$$

and

$$\mathbf{H} \;\to\; \mathbf{H}^a \quad \text{as } |\underline{x}| \to \infty.$$

On the boundary of the superconductor, $\partial\Omega$, we have that

$$[\mathbf{H}_\tau] = [\mathbf{H}_\nu] = 0,$$

where $[\mathbf{H}_\tau]$ and $[\mathbf{H}_\nu]$ denote the jumps in the tangential and normal components, respectively, of $\mathbf{H}$ across $\partial\Omega$.

In order to consider homogeneous boundary conditions at infinity, it is convenient to introduce a current density $\mathbf{J}^e$ defined by

$$\mathbf{J}^e \;=\begin{cases} \mathbf{0} & \text{in } \Omega, \\ \mathbf{J}^a & \text{in } \mathbb{R}^2/\overline{\Omega}. \end{cases}$$

Associated with $\mathbf{J}^e$ is the magnetic field $\mathbf{H}^e$ such that

$$\begin{aligned} \text{curl } \mathbf{H}^e &= \mathbf{J}^e \quad \text{in } \mathbb{R}^2, \\ \nabla \cdot \mathbf{H}^e &= 0 \quad \text{in } \mathbb{R}^2, \\ \mathbf{H}^e &\to \mathbf{H}^a \quad \text{as } |\underline{x}| \to \infty. \end{aligned}$$

Finally, we use the shift

$$\hat{\mathbf{J}} = \mathbf{J} - \mathbf{J}^e \quad \text{and} \quad \hat{\mathbf{H}} = \mathbf{H} - \mathbf{H}^e$$

to give the problem

$$\frac{\partial \hat{\mathbf{H}}}{\partial t} + \text{curl } (\rho \hat{\mathbf{J}}) = -\frac{\partial \mathbf{H}^e}{\partial t} \qquad \text{in } \Omega, \tag{2.1}$$

$$\text{curl } \hat{\mathbf{H}} = \hat{\mathbf{J}} \qquad \text{in } \mathbb{R}^2, \tag{2.2}$$

$$\nabla \cdot \hat{\mathbf{H}} = 0 \qquad \text{in } \mathbb{R}^2, \tag{2.3}$$

$$|\hat{\mathbf{J}}| \leq J_c \qquad \text{in } \Omega \tag{2.4}$$

together with the boundary condition

$$\hat{\mathbf{H}} \to \mathbf{0} \text{ as } |\underline{x}| \to \infty.$$

Note that interpreting (2.2), (2.3) in conservation form yields the compatibility boundary conditions

$$\left[\hat{\mathbf{H}}_\nu\right] = \left[\hat{\mathbf{H}}_\tau\right] = 0 \quad \text{on } \partial\Omega.$$

It follows by the assumption $\mathbf{J}^a = J^a(\underline{x}, t)\mathbf{e}_3$ and the definitions of $\mathbf{J}$ and $\mathbf{J}^e$ that $\hat{\mathbf{J}} = (0, 0, J)$, where $J \in K$. From this last set of equations and using the assumption that $\hat{\mathbf{H}}$ lies in the (x_1, x_2) plane, we see that there exists a scalar potential $q(\underline{x}, t)$, $\underline{x} \in \mathbb{R}^2$, for $\hat{\mathbf{H}}$ such that $\hat{\mathbf{H}} = (\nabla^\perp q, 0)$.

Furthermore, q satisfies

$$-\Delta q = J \quad \text{in } \mathbb{R}^2 \tag{2.5a}$$

and

$$|\nabla^\perp q| \to 0 \quad \text{as } |\underline{x}| \to \infty. \tag{2.5b}$$

Imposing the condition

$$\int_\Omega q d\underline{x} = 0, \tag{2.5c}$$

the problem (2.5a)–(2.5c) is known to have a unique solution, which we denote by

$$q = GJ.$$

Similarly, there exists a scalar potential q^e for $\mathbf{H}^e$, unique up to a constant function in time, such that

$$\mathbf{H}^e = (\nabla^\perp q^e, 0), \quad \nabla^\perp q^e \to \underline{H}^a \quad \text{as } |\underline{x}| \to \infty.$$

We may rewrite (2.1) in the form

$$\begin{aligned} \nabla^\perp \left(\frac{\partial q}{\partial t} + \rho J \right) &= -\nabla^\perp \frac{\partial q^e}{\partial t} \\ \Rightarrow \nabla^\perp \left(\frac{\partial GJ}{\partial t} + \rho J \right) &= -\nabla^\perp \frac{\partial q^e}{\partial t}. \end{aligned}$$

Hence, fixing q^e, we obtain

$$\frac{\partial GJ}{\partial t} + \rho J - \lambda(t) = -\frac{\partial q^e}{\partial t} := f,$$

where λ is an arbitrary function of time.

Multiplying the above equation by $\eta - J$ for $\eta \in K$, integrating over Ω, and using the fact that $(1, \eta - J) = 0$, we have

$$\left(\frac{\partial GJ}{\partial t}, \eta - J \right) = (f, \eta - J) - (\rho J, \eta - J).$$

Since $\rho(r) \in \beta(|J|)$ and $|\eta| \leq J_c$, we have

$$(\rho J, \eta - J) \leq 0.$$

Hence, we obtain problem $(\mathbf{P})$.

The above formulation of Bean's model is the basis of the numerical algorithm proposed by Prigozhin in [9, 11] using an explicit formula for the integral operator G. The discretization is then based upon piecewise constant finite elements. This approach leads to a dense matrix. In the following we use the finite element method to approximate G but never form the matrix associated with this finite element approximation. Whenever G is required we use an elliptic solve. In this paper an error bound is proved and an iterative method is proposed for the resulting discrete variational inequality. For an engineering application of $(\mathbf{P})$, see [2, 3].

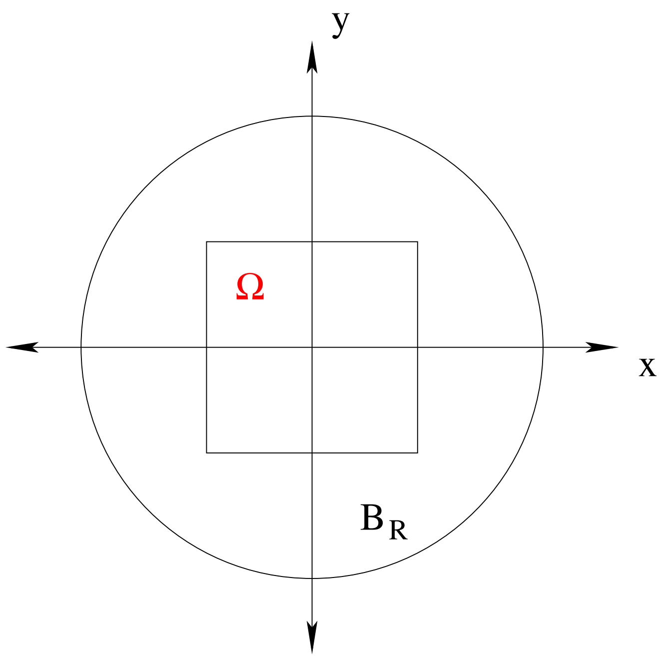

FIG. 2.1. *Reduction in the domain of the problem.*

2.2. Reduction to a bounded domain. From a computational viewpoint, discretizing the whole of $\mathbb{R}^2$ in order to find the operator G is not practical. A natural approach is to restrict the problem to a large bounded region B_R containing Ω and to write an exact boundary condition for Gv on ∂B_R.

Consider the situation where Ω is embedded in a large circle B_R of radius R; see Figure 2.1.

We consider a Dirichlet-to-Neumann mapping which relies on the harmonic property of Gv outside B_R and the boundedness of $\nabla^{\perp} Gv$ in $L^2(\mathbb{R}^2)$. This method of truncating a problem defined on an infinite domain to one defined on a finite domain is described in [6]. An overview is given here.

For $w \in H^{1/2}(\partial B_R)$ let z solve

$$-\Delta z = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{B_R}, \tag{2.6}$$

$$z = w \quad \text{on} \quad \partial B_R, \tag{2.7}$$

$$\nabla z \in L^2(\mathbb{R}^2 \backslash \overline{B_R}). \tag{2.8}$$

It follows that we have a Fourier expansion

$$z(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\theta) + b_k \sin(k\theta)) R^k r^{-k},$$

where a_k, b_k are the Fourier coefficients for $w = w(\theta)$ on ∂B_R.

Differentiating with respect to r and letting $r \to R$ gives

$$\frac{\partial z}{\partial r}(R,\theta) = -\sum_{k=1}^{\infty} \frac{k}{R} (a_k \cos(k\theta) + b_k \sin(k\theta)). \tag{2.9}$$

Since

$$a_k = -\frac{1}{k\pi}\int_0^{2\pi} \frac{\partial w}{\partial \varphi}\sin(k\varphi)d\varphi \quad \text{and} \quad b_k = \frac{1}{k\pi}\int_0^{2\pi} \frac{\partial w}{\partial \varphi}\cos(k\varphi)d\varphi,$$

substituting into (2.9) gives the relation

$$\left.\frac{\partial z}{\partial r}\right|_{\partial B_R}(\theta) = \mathcal{B}(w)(\theta) := -\sum_{k=1}^{\infty}\frac{1}{R\pi}\int_0^{2\pi}\frac{\partial w}{\partial \varphi}\sin(k(\varphi-\theta))d\varphi. \tag{2.10}$$

Let z be a solution of (2.6)–(2.8) for w being the trace of Gv on ∂B_R. Taking $\mathcal{B}(\cdot)$ to be defined as above, it follows that Gv solves the following Neumann problem defined on B_R:

$$-\Delta Gv = v \quad \text{in } B_R, \frac{\partial Gv}{\partial \nu} = \mathcal{B}(Gv) \quad \text{on } \partial B_R. \tag{2.11}$$

Multiplying (2.11) by a test function $\eta \in H^1(B_R)$, integrating over B_R, and then integrating by parts yield the equivalent variational problem:

For $v \in \mathcal{F}_0$, find $Gv \in H^1(B_R)$ such that

$$(Gv, 1) = 0, \quad a(Gv, \eta) + b(Gv, \eta) = (v, \eta) \quad \forall\, \eta \in H^1(B_R), \tag{2.12}$$

where for $\xi, \eta \in H^1(B_R)$,

$$a(\xi,\eta) := \int_{B_R} \nabla\xi\cdot\nabla\eta\; d\underline{x} \quad \text{and} \quad b(\xi,\eta) := \int_{\partial B_R}\mathcal{B}(\xi)\eta\; dS.$$

The existence of a unique solution Gv to this variational problem is easily proved.

We define

$$A(\xi,\eta) := a(\xi,\eta) + b(\xi,\eta) \quad \forall\, \xi,\eta \in H^1(B_R) \tag{2.13}$$

together with the seminorm and norm

$$|\eta|_A^2 := A(\eta,\eta) \quad \forall\, \eta \in H^1(B_R), \quad ||\eta||_{A^{-1}}^2 := |G\eta|_A^2 \quad \forall\, \eta \in \mathcal{F}_0. \tag{2.14}$$

Henceforth we define the L^2 norm and the H^1 norm and seminorm over X respectively by

$$||\eta||_{0,X}^2 = \int_X |\eta|^2 d\underline{x}, \quad ||\eta||_{1,X}^2 = \int_X (|\eta|^2 + |\nabla\eta|^2)d\underline{x} \quad \text{and} \quad |\eta|_{1,X}^2 = \int_X |\nabla\eta|^2 d\underline{x}.$$

From [6] we have that A is continuous with respect to the H^1 norm; that is, for all $\xi, \eta \in H^1(B_R)$

$$|A(\xi,\eta)| \;\le\; C||\xi||_{1,B_R}||\eta||_{1,B_R}. \tag{2.15}$$

Using (2.12)–(2.15), we have the following useful result:

$$(\xi,\eta) = A(G\xi,\eta) \le |G\xi|_A|\eta|_A \le C||\xi||_{A^{-1}}||\eta||_{1,B_R} \quad \forall\, \eta \in H^1(B_R),\ \xi \in \mathcal{F}_0. \tag{2.16}$$

3. Finite element approximation. In this section we consider a finite element approximation of (**P**) under the following assumptions on the partitioning:

(A) Let Ω be a polygon and let T_h^1 be a quasi-uniform partitioning of Ω into disjoint open simplices κ with $h_\kappa := \mathrm{diam}(\kappa)$ and $h := \max_{\kappa\in T_h^1} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa\in T_h^1}\overline{\kappa}$.

(B) Let T_h^2 be a partitioning of B_R into disjoint open elements $\kappa \in T_h^2$ such that
- $\cup_{\kappa\in T_h^2}\overline{\kappa} = \overline{B}_R$,
- either $\kappa\cap\Omega$ is empty or $\kappa \in T_h^1$,
- if $\overline{\kappa}\cap\partial B_R = \emptyset$, or a point, then κ is a simplex; otherwise, κ is a three-sided element with a curved edge on ∂B_R.

Associated with T_h^1 is the finite element space of continuous piecewise linear functions on Ω such that

$$S_h^1 = \left\{\chi \in C(\overline{\Omega}) : \chi|_\kappa \quad \text{is linear} \ \forall\ \kappa \in T_h^1\right\} \subset H^1(\Omega).$$

Similarly associated with T_h^2 is the finite element space of continuous functions on B_R such that

$$S_h^2 = \left\{\chi \in C(\overline{B_R}) : \chi|_\kappa \quad \text{is linear} \ \ \forall\ \kappa \in T_h^2\right\} \subset H^1(B_R).$$

The discrete inner product $(\cdot,\cdot)^h$ is defined by numerical integration in the following way.

Associated with each node $\underline{x}_i$, $i = 1, 2, \ldots, M$, of S_h^1 we have a lumped mass matrix value $\mathcal{M}_i > 0$. We now introduce a discrete semi-inner product on $L^2(\Omega)$, defined by

$$(\eta_1,\eta_2)^h := \int_\Omega \Pi^h(\eta_1\eta_2)d\underline{x} = \sum_{i=1}^{M}\mathcal{M}_i(\eta_1\eta_2)(\underline{x}_i), \tag{3.1}$$

where $\Pi^h : C(\overline{\Omega}) \to S_h^1$ is the standard linear interpolation operator.

We introduce the $L^2(\Omega)$ projection operator $Q^h : L^2(\Omega) \to S_h^1$ such that

$$\left(Q^h\eta,\chi\right)^h = (\eta,\chi) \quad \ \forall\ \chi \in S_h^1. \tag{3.2}$$

Similar to (2.12) we introduce the operator $G^h : \mathcal{F}_0 \to \mathcal{V}_h := \{v_h \in S_h^2 : (v_h, 1) = 0\}$ such that

$$A(G^h\xi,\chi) = (Q^h\eta,\chi)^h \quad \ \forall\ \xi \in \mathcal{F}_0,\ \chi \in S_h^2, \tag{3.3}$$

and we define the norm

$$||\eta||_{A^{-h}}^2 := \left|G^h\eta\right|_A^2 \quad \forall\eta \in \mathcal{F}_0.$$

It follows from (3.2) and (3.3) similarly to (2.16) that

$$(\xi,\chi) \ \leq\ C||\xi||_{A^{-h}}||\chi||_{1,B_R} \quad \ \forall\ \chi \in S_h^2,\ \xi \in \mathcal{F}_0. \tag{3.4}$$

From [6] we have the following useful results:

$$\left\|(G - G^h)\eta\right\|_{0,B_R} \ \leq\ C_R h^2 \left\|\eta\right\|_{0,\Omega} \qquad \forall\ \eta \in \mathcal{F}_0, \tag{3.5}$$

$$\left|(G - G^h)\eta\right|_{1,B_R} \leq C_R h \, \|\eta\|_{0,\Omega} \qquad \forall\, \eta \in \mathcal{F}_0, \tag{3.6}$$

$$\left|G^h \chi\right|_A \leq |G\chi|_A \qquad \forall\, \chi \in S_h^1, \tag{3.7}$$

and using (3.6) it follows that

$$|G\chi|_A \leq C\left|G^h\chi\right|_A \qquad \forall\, \chi \in S_h^1. \tag{3.8}$$

Lastly from (2.16) and an inverse inequality we have the following for all $\chi \in \mathcal{F}_0 \cap S_h^2$:

$$\|\chi\|_{0,\Omega}^2 \leq C \, |G\chi|_A ||\chi||_{1,\Omega} \leq Ch^{-1}|G\chi|_A \, \|\chi\|_{0,\Omega}$$

$$\Rightarrow \|\chi\|_{0,\Omega} \leq Ch^{-1}|G\chi|_A. \tag{3.9}$$

LEMMA 3.1. *We have*

$$\left|G(\eta - Q^h\eta)\right|_A \leq Ch \, \|\eta\|_{0,\Omega} \qquad \forall\, \eta \in \mathcal{F}_0. \tag{3.10}$$

Proof. Using (2.12), (3.2), (3.3), (3.5), Hölder's inequality, and the well-known estimate

$$\left|(\xi, \eta) - (\xi, \eta)^h\right| \leq Ch^2|\xi|_{1,\Omega}|\eta|_{1,\Omega} \leq Ch|\xi|_{1,\Omega} \, \|\eta\|_{0,\Omega} \quad \forall\, \xi, \eta \in S_h^1, \tag{3.11}$$

we have the following for all $\eta \in \mathcal{F}_0$:

$$\begin{aligned}
\left|G(\eta - Q^h\eta)\right|_A^2 &= A(G(\eta - Q^h\eta), G(\eta - Q^h\eta)) \\
&= (G(\eta - Q^h\eta), \eta - Q^h\eta) \\
&= ((G - G^h)(\eta - Q^h\eta), \eta - Q^h\eta) \\
&\quad + (G^h(\eta - Q^h\eta), Q^h\eta)^h - (G^h(\eta - Q^h\eta), Q^h\eta) \\
&\leq \left\|(G - G^h)(\eta - Q^h\eta)\right\|_{0,\Omega} \left\|\eta - Q^h\eta\right\|_{0,\Omega} \\
&\quad + Ch\left|G^h(\eta - Q^h\eta)\right|_{1,\Omega} \left\|Q^h\eta\right\|_{0,\Omega} \\
&\leq Ch^2 \left\|\eta - Q^h\eta\right\|_{0,\Omega}^2 + Ch\left|G^h(\eta - Q^h\eta)\right|_A \left\|Q^h\eta\right\|_{0,\Omega}.
\end{aligned}$$

The result follows by noting (3.7) and using Young's inequality. ◻

Finally we introduce a finite element approximation of (**P**):

($\mathbf{P}_h$) Find $J_h \in K_h$ such that $J_h(\cdot, 0) = Q^h J_0$ and

$$\left(\frac{\partial}{\partial t} G^h J_h, \chi - J_h\right) \geq (f, \chi - J_h) \qquad \forall\, \chi \in K_h, \tag{3.12}$$

where

$$K_h := \left\{\chi \in S_h^1 : |\chi| \leq J_c, \quad (\chi, 1) = 0\right\}.$$

Remark 3.1. Let the assumptions (A) hold. Then there exists a unique solution J_h to ($\mathbf{P}_h$) such that

$$||J_h||_{L^\infty(0,T;L^\infty(\Omega))} + \left\|\frac{\partial}{\partial t} G J_h\right\|_{L^\infty(0,T;A)} \leq C. \tag{3.13}$$

LEMMA 3.2. *The unique solutions of* $(\mathbf{P}_h)$ *and* $(\mathbf{P}_h)$ *satisfy*

$$||J - J_h||^2_{L^\infty(0,T;A^{-1})} \le Ch. \tag{3.14}$$

Proof. Since $J_h \in K$ using (1.1), (2.16), and (3.12) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|J - J_h\|^2_{A^{-1}} &= \left(\frac{\partial}{\partial t}G(J - J_h), J - J_h\right) \\
&\le (f, J - J_h) - \left(\frac{\partial}{\partial t}GJ_h, J - J_h\right) \\
&= (f, J - J_h) - \left(\frac{\partial}{\partial t}G^h J_h, Q^h J - J_h\right) - \left(\frac{\partial}{\partial t}G^h J_h, J - Q^h J\right) \\
&\quad - \left(\frac{\partial}{\partial t}(G - G^h)J_h, J - J_h\right) \\
&\le (f, J - J_h) - \big(f, Q^h J - J_h\big) - \left(\frac{\partial}{\partial t}G^h J_h, J - Q^h J\right) \\
&\quad - \left(\frac{\partial}{\partial t}(G - G^h)J_h, J - J_h\right) \\
&= \left(f - \frac{\partial}{\partial t}G^h J_h, J - Q^h J\right) - \left(\frac{\partial}{\partial t}(G - G^h)J_h, J - J_h\right) \\
&\le \left|f - \frac{\partial}{\partial t}G^h J_h\right|_A \|J - Q^h J\|_{A^{-1}} + \left\|\frac{\partial}{\partial t}(G - G^h)J_h\right\|_{0,\Omega} \|J - J_h\|_{0,\Omega}.
\end{aligned}$$

Using the above inequality together with (1.4), (3.9), (3.5), (3.10), and (3.13) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|J - J_h\|^2_{A^{-1}} &\le Ch + Ch^2 \left\|\frac{\partial}{\partial t}J_h\right\|_{0,\Omega} \|J - J_h\|_{0,\Omega} \\
&\le Ch + Ch\left|\frac{\partial}{\partial t}GJ_h\right|_A.
\end{aligned}$$

Integrating from 0 to t and using (3.13) gives the required result. ∎

Remark 3.2. This is a suboptimal error bound because of the error term $(f - \frac{\partial}{\partial t}G^h J_h, J - P^h J)$, arising due to the variational inequality, which only gives $\mathcal{O}(h)$ because of the lack of H^1 regularity of J.

4. Fully discrete model. In this section we consider a fully discrete discretization of $(\mathbf{P})$. Setting $N\Delta t = T$ and $t_n : n\Delta t$ for $n = 0 \to N$ and for any $\chi_h \in S^1_h$, $n = 0, 1, \ldots,$ we set

$$\delta_t \chi^n = \frac{\chi^n - \chi^{n-1}}{\Delta t}.$$

We consider the following fully discrete discretization of $(\mathbf{P})$:

$(\mathbf{P}_{h,\Delta t})$ For $n = 1 \to N$, find $J^n_h \in K_h$ such that $J^0_h = Q^h J_0$ and

$$\big(G^h(\delta_t J^n_h), \chi - J^n_h\big) \ge \big(f^n, \chi - J^n_h\big) \quad \forall\ \chi\ \in K_h, \tag{4.1}$$

where $f^n := f(\cdot, t_n)$.

LEMMA 4.1. *Let the assumptions* (A) *hold. Then for* $n = 1 \to N$ *there exists a unique solution* J_h^n *to* $(\mathbf{P}_{h,\Delta t})$ *such that*

$$\max_{n=1\to N} ||\delta_t J_h^n||^2_{A^{-h}} \leq C. \tag{4.2}$$

Proof. Existence and uniqueness for (4.1) are standard. Setting $\chi = J_h^{n-1}$ in (4.1), dividing by Δt, and noting (1.4) and (3.4) gives

$$\left(G^h \delta_t J_h^n, \delta_t J_h^n\right) \leq (f^n, \delta_t J_h^n)$$

$$\Rightarrow ||\delta_t J_h^n||^2_{A^{-h}} \leq ||f^n||_{1,B_R} ||\delta_t J_h^n||_{A^{-h}}$$

which together with (1.4) yields (4.2). ◻

Before we derive an error bound on the solutions of $(\mathbf{P}_h)$ and $(\mathbf{P}_{h,\Delta t})$ we introduce some useful notation. For $n \geq 1$ we set

(4.3)

$$J_{h,\Delta t}(t) := \frac{t - t_{n-1}}{\Delta t} J_h^n + \frac{t_n - t}{\Delta t} J_h^{n-1}, \quad f_{\Delta t}(t) := \frac{t - t_{n-1}}{\Delta t} f^n + \frac{t_n - t}{\Delta t} f^{n-1} \quad \forall t \in [t_{n-1}, t_n],$$

and

$$\hat{J}_{h,\Delta t}(t) := J_h^n, \quad \hat{f}_{\Delta t}(t) := f^n \quad \forall\ t \in (t_{n-1}, t_n]. \tag{4.4}$$

From (4.3) and (4.4) it follows that for a.e. $t \in (0, T)$,

$$J_{h,\Delta t} - \hat{J}_{h,\Delta t} = -(t_n - t)\frac{\partial}{\partial t} J_{h,\Delta t}, \quad f_{\Delta t} - \hat{f}_{\Delta t} = -(t_n - t)\frac{\partial}{\partial t} f_{\Delta t}. \tag{4.5}$$

We also introduce for $t \in (0, T)$,

$$\mathcal{R}(t) := \left(\hat{f}_{\Delta t} - \frac{\partial}{\partial t} G^h J_{h,\Delta t}, \hat{J}_{h,\Delta t} - J_{h,\Delta t}\right)$$

$$= (t_n - t)\left(\hat{f}_{\Delta t} - \frac{\partial}{\partial t} G^h J_{h,\Delta t}, \frac{\partial}{\partial t} J_{h,\Delta t}\right), \quad t \in (t_{n-1}, t_n], \tag{4.6}$$

and for $t \in (0, T]$,

$$\mathcal{D}(t) := \mathcal{D}^n := -\left(G^h(\delta_t J_h^n), \delta_t J_h^n\right) + \left(G^h(\delta_t J_h^{n-1}), \delta_t J_h^n\right), \quad t \in (t_{n-1}, t_n], \tag{4.7}$$

with J_h^{-1} satisfying (4.1) and

$$\left|\left|\frac{J^0 - J^{-1}}{\Delta t}\right|\right|^2_{A^{-h}} = \left(G^h(\delta_t J_h^0), \delta_t J_h^0\right) \leq C. \tag{4.8}$$

LEMMA 4.2. *For a.e.* $t \in (0, T)$ *we have that*

$$\mathcal{R}(t) \leq (t_n - t)\left[\mathcal{D}(t) + \Delta t\left(\frac{\partial}{\partial t} f_{h,\Delta t}, \frac{\partial}{\partial t} J_{h,\Delta t}\right)\right], \quad t \in (t_{n-1}, t_n], \tag{4.9}$$

and

$$\int_0^T \mathcal{R}(t)dt \le C(\Delta t)^2. \tag{4.10}$$

Proof. Setting $\chi = J_h^n$ in (4.1) for $n = n-1$ and using the definitions of $\mathcal{D}(t)$ and $\mathcal{R}(t)$, we have

$$\begin{aligned}\mathcal{R}(t) &= -(t_n - t)\left(\frac{\partial}{\partial t}G^h J_{h,\Delta t}, \frac{\partial}{\partial t}J_{h,\Delta t}\right) + (t_n - t)\left(\hat{f}_{\Delta t}(t) - \hat{f}_{\Delta t}(t-\Delta t), \frac{\partial}{\partial t}J_{h,\Delta t}\right)\\ &\quad + (t_n - t)\left(\hat{f}_{\Delta t}(t-\Delta t), \frac{\partial}{\partial t}J_{h,\Delta t}\right)\\ &\le (t_n - t)\mathcal{D}^n + (t_n - t)\left(\hat{f}_{\Delta t}(t) - \hat{f}_{\Delta t}(t-\Delta t), \frac{\partial}{\partial t}J_{h,\Delta t}\right),\end{aligned}$$

and (4.9) follows by using (4.3). We now integrate (4.9) from 0 to t and use (1.4), (3.4), and (4.2) to obtain

$$\int_0^T \mathcal{R}(t)dt = \sum_{n=1}^{N} \mathcal{D}^n \int_{t_{n-1}}^{t_n} (t_n - t)dt + \int_0^T \Delta t\left(\frac{\partial}{\partial t}f_{\Delta t}, \frac{\partial}{\partial t}J_{h,\Delta t}\right)(t_n - t)dt$$

$$\le \sum_{n=1}^{N} \frac{(\Delta t)^2}{2}\mathcal{D}^n + (\Delta t)^2 \int_0^T \left|\left|\frac{\partial}{\partial t}f_{\Delta t}\right|\right|_{1,B_R}\left|\left|\frac{\partial}{\partial t}J_{h,\Delta t}\right|\right|_{A^{-h}} dt \tag{4.11}$$

$$\le \sum_{n=1}^{N} \frac{(\Delta t)^2}{2}\mathcal{D}^n + C(\Delta t)^2. \tag{4.12}$$

To bound the first term on the right-hand side we use the identity

$$2\big(G^h(a-b), a\big) = \big(G^h a, a\big) - \big(G^h b, b\big) + \big(G^h(a-b), a-b\big)$$

to obtain

$$2\mathcal{D}^n \le \big(G^h(\delta_t J_h^{n-1}), \delta_t J_h^{n-1}\big) - \big(G^h(\delta_t J_h^n), \delta_t J_h^n\big).$$

Summing the above inequality from $n = 1 \to N$ and using (3.4) and (4.8), we have

$$2\sum_{n=1}^{N} \mathcal{D}^n \le \big(G^h(\delta_t J_h^0), \delta_t J_h^0\big) - \big(G^h(\delta_t J_h^N), \delta_t J_h^N\big)$$

$$\le \big(G^h(\delta_t J_h^0), \delta_t J_h^0\big) \le C. \tag{4.13}$$

Using (4.13) in (4.12), we conclude (4.10). ∎

LEMMA 4.3. *The unique solutions of* $(\mathbf{P}_h)$ *and* $(\mathbf{P}_{h,\Delta t})$ *satisfy*

$$||J_h - J_{h,\Delta t}||_{L^\infty(0,T;A^{-1})} \le C\Delta t. \tag{4.14}$$

Proof. Setting $\chi = J_h$ in (4.1) and $\chi = J_{h,\Delta t}$ in (3.12) and adding the resulting inequalities gives

$$\left(\frac{\partial}{\partial t} G^h(J_{h,\Delta t} - J_h), J_{h,\Delta t} - J_h\right) \leq \left(\hat{f}_{\Delta t} - f, J_{h,\Delta t} - J_h\right) + \left(\frac{\partial}{\partial t} G^h J_{h,\Delta t} - \hat{f}_{\Delta t}, J_{h,\Delta t} - \hat{J}_{h,\Delta t}\right).$$

Noting (3.4) and (4.6), we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}||J_{h,\Delta t} - J_h||^2_{A^{-h}} \leq \left|\left|\hat{f}_{\Delta t} - f\right|\right|_{1,B_R} ||J_{h,\Delta t} - J_h||_{A^{-h}} + \mathcal{R}.$$

From Lemma 3.6 in [8] we conclude that

$$\max_{t\in[0,T]} ||J_{h,\Delta t} - J_h||_{A^{-h}} \leq \left(||J_{h,\Delta t}(0) - J_h(0)||^2_{A^{-h}} + \int_0^T \mathcal{R}(t)dt\right)^{1/2} + \int_0^T \left|\left|\hat{f}_{\Delta t} - f\right|\right|_{1,B_R} dt,$$

and noting (1.4), (3.8), (4.4), and (4.10) yields the required result. ∎

Finally we have our main result.

THEOREM 4.4. *Let the assumptions* (A) *hold. Then the unique solutions* $\{J_h^n\}_{n=0}^N$ *to* $(\mathbf{P}_{h,\Delta t})$ *and* J *to* $(\mathbf{P})$ *satisfy*

$$||J - J_{h,\Delta t}||_{L^\infty(0,T;A^{-1})} \leq C(T)(h^{1/2} + \Delta t).$$

Proof. The desired result follows directly from (3.14) and (4.14). ∎

Recalling that $\hat{\underline{H}} = \nabla^\perp GJ$ and setting $\hat{\underline{H}}_h^n = \hat{\underline{H}}_h(t_n) := \nabla^\perp G^h J_h(t_n)$ and $\hat{\underline{H}}_{h,\Delta t}$ as in (4.3), we conclude the following.

COROLLARY 4.1. *The error between the magnetic field* $\underline{H}$ *and its approximation* $\underline{H}_{h,\Delta t}$ *is*

$$\left\|\underline{H} - \underline{H}_{h,\Delta t}\right\|_{L^\infty(0,T;L^2(B_R))} \leq C(T)(h^{1/2} + \Delta t).$$

5. Algorithm for solving $(\widehat{\mathbf{P}}_{\mathbf{h},\Delta\mathbf{t}})$. In the numerical simulations presented in section 6 we solve the following approximation of $(\widehat{\mathbf{P}}_{h,\Delta t})$:

$(\widehat{\mathbf{P}}_{h,\Delta t})$ For $n = 1 \to N$, find $J_h^n \in K_h$ such that $J_h^0 = Q^h J_0$ and

$$\left(\hat{G}^h\left(\delta_t \widehat{J}_h^n\right), \chi - \widehat{J}_h^n\right)^h \geq \left(f^n, \chi - \widehat{J}_h^n\right)^h \quad \forall\, \chi \in K_h, \tag{5.1}$$

where $f^n := f(\cdot, t_n)$ and the operator $\hat{G}^h : \mathcal{V}_h \to \mathcal{V}_h$ is such that

$$A(\hat{G}^h\xi, \chi) = (\xi, \chi)^h \quad \forall\, \xi \in \mathcal{V}_h, \chi \in S_h^2.$$

Below we give an algorithm for solving $(\widehat{\mathbf{P}}_{h,\Delta t})$. See [5] for an account of iterative methods for solving discrete variational inequalities.

Reformulating $(\widehat{\mathbf{P}}_{h,\Delta t})$ gives the following problem:

Given $J_h^0 = P^h J_0$, for $n = 1 \to N$, find $J_h^n \in K_h$ and $\lambda^n \in \mathbb{R}$ such that $|J_h^n| \le J_c$, $(J_h^n, 1)^h = 0$, and

$$\left(\hat{G}^h \widehat{J}_h^n, \chi - \widehat{J}_h^n\right)^h \ge \left(\Delta t f^n + \lambda^n + \hat{G}^h \widehat{J}_h^{n-1}, \chi - \widehat{J}_h^n\right)^h \quad \forall\, \chi \in S_h^1 \text{ such that } |\chi| \le J_c.$$

Setting $\Lambda_h^n := \Delta t f^n + \hat{G}^h \widehat{J}_h^{n-1}$, the above problem is equivalent to the following problem:

Find $\widehat{J}_h^n \in S_h^1$ such that $(J_h^n, 1)^h = 0$ and

$$\hat{G}^h \widehat{J}_h^n - \Lambda_h^n - \lambda^n + \beta_h^n = 0$$

$$\Leftrightarrow \frac{1}{\mu} \widehat{J}_h^n + \hat{G}^h \widehat{J}_h^n - \lambda^n + \beta_h^n = \Lambda_h^n + \frac{1}{\mu} \widehat{J}_h^n, \tag{5.2}$$

where $\beta_h^n(\underline{x}_i) \in \beta(J_h^n(\underline{x}_i))$.

We solve (5.2) iteratively using a splitting algorithm of Lions and Mercier [7]. Let $\widehat{J}_h^0$ be given; for fixed μ we construct $J_h^{n,k+1}$, $\beta_h^{n,k+1}$, and $\lambda^{n,k+1}$ iteratively by solving for $k \ge 0$:

$$\frac{1}{\mu} \widehat{J}_h^{n,k+1/2} + \hat{G}^h \widehat{J}_h^{n,k+1/2} = \Lambda_h^n + \frac{1}{\mu} \widehat{J}_h^{n,k} - \beta_h^{n,k} + \lambda^{n,k} := \tilde{\Lambda}_h^{n,k}, \tag{5.3}$$

$$\frac{1}{\mu} \widehat{J}_h^{n,k+1} - \lambda^{n,k+1} + \beta_h^{n,k+1} = \Lambda_h^n + \frac{1}{\mu} \widehat{J}_h^{n,k+1/2} - \hat{G}^h \widehat{J}_h^{n,k+1/2} := F^{n,k+1/2}, \tag{5.4}$$

$$(J_h^{n,k+1}, 1)^h = 0,$$

where $\beta_h^{n,k+1}(\underline{x}_i) \in \beta(J_h^{n,k+1}(\underline{x}_i))$. To solve (5.3) we use (3.3) to rewrite

$$\frac{1}{\mu}\left(\bar{J}_h^{k+1/2}, \chi\right)^h + \left(\hat{G}^h \bar{J}_h^{k+1/2}, \chi\right)^h = \left(\tilde{\Lambda}_h^{n,k}, \chi\right)^h$$

as

$$\frac{1}{\mu} A(\hat{G}^h \bar{J}_h^{k+1/2}, \chi) + \left(\hat{G}^h \bar{J}_h^{k+1/2}, \chi\right)^h = \left(\tilde{\Lambda}_h^{n,k}, \chi\right)^h, \tag{5.5}$$

where $\bar{J}_h^{k+1/2} = \widehat{J}_h^{k+1/2} - f^n$.

At the ith node we may rewrite (5.4) using the projection

$$J_i^{n,k+1} = P(\mu(F_i^{n,k+1/2} + \lambda^{n,k+1})), \tag{5.6}$$

where

$$P(r) = \begin{cases} J_c & \text{if} \quad r \ge J_c, \\ r & \text{if} \quad |r| < J_c, \\ -J_c & \text{if} \quad r \le -J_c. \end{cases}$$

Noting that $(J_h^{n,k+1}, 1)^h = 0$, $\lambda^{n,k+1}$ solves the equation

$$g(\lambda) = \sum_i \mathcal{M}_i P(\mu(F_i^{n,k+1/2} + \lambda)) = 0. \tag{5.7}$$

To obtain the solution at the $(k+1)$th time step we proceed as follows:

Step 1. Solve (5.5) to obtain $\hat{G}^h \bar{J}_h^{k+1/2}$.

Step 2. Set $\hat{G}^h J_h^{n,k+1/2} = \hat{G}^h \bar{J}_h^{k+1/2} + f^n$.

Step 3. Use (5.3) to obtain $\widehat{J}_h^{n,k+1/2}$.

Step 4. Solve (5.6) and (5.7) to obtain $\widehat{J}_h^{n,k+1}$.

Step 5. Use (5.4) to obtain $\beta_h^{n,k+1}$.

Step 6. If $|\widehat{J}_h^{n,k+1} - \widehat{J}_h^{n,k}| \leq \text{tol}$, then set $\widehat{J}_h^n = \widehat{J}_h^{n,k+1}$; else set $\widehat{J}_h^{n,k} = \widehat{J}_h^{n,k+1/2}$ and go to Step 1.

The above procedure is relatively cheap apart from Step 1, which involves the solution of a large sparse matrix problem,

$$A\mathbf{x} = \mathbf{f}, \quad A \in \mathbb{R}^{N \times N}.$$

In general N is required to be large, so that interfaces between critical current and noncritical current can be captured.

Since the matrix A remains fixed throughout time, we could calculate the inverse, or an LU decomposition, of A at the beginning. Due to the nonlocal boundary condition the LU decomposition of this matrix produces $O(N^{3/2})$ entries, and thus, for large problems this is not practical.

Since Step 1 is part of an iteration, we need not solve this problem exactly. In the following section ten or fewer preconditioned GMRES iterations (see [12]) are used with an ILU decomposition used as a preconditioner. This allows large problems to be solved and accurate solutions to be obtained.

Note that Step 4 is well defined for $J_h^{n,k+1}$. It is easily seen that the function g is continuous and monotone piecewise linear which takes negative values for sufficiently negative λ and positive values for sufficiently positive λ, and hence (5.7) has a solution. Furthermore it has only a nonunique solution when $g(\lambda) = 0$ in an interval and in such an interval we observe that $P(F_i^{n,k+1/2} + \lambda)$ is constant for each i; hence the solution of (5.6) is unique. A solution of (5.7) can be found by efficiently by using the bisection method.

In [9, 11] Prigozhin solves the discrete variational inequality associated with the full matrix approximation of G using a projected SOR algorithm. We avoid doing this by using the splitting algorithm defined above in which it is not necessary to form the solution operator G explicitly but its action is calculated by the use of an elliptic solve. That is, (5.3) is implemented using elliptic solve (5.5). The constraint condition is then handled by (5.4), which is easily solved by the projection (5.6) and the Lagrange multiplier equation (5.7).

In practice we do not actually compute $G^h J_h$. Instead we approximate it by replacing the nonlocal boundary inner product $b(\cdot,\cdot)$ with a truncated version $b_M(\cdot,\cdot)$, where

$$b_M(\xi,\eta) = \int_{\partial\Omega} \mathcal{B}_M(\xi)\eta dS$$

with

$$\mathcal{B}_M(w)(\theta) := \int_{\partial\Omega} \sum_{k=1}^{M} \frac{1}{R\pi} \int_0^{2\pi} \frac{\partial w}{\partial \varphi} \sin\left(k(\varphi - \theta)\right) d\varphi.$$

Error analysis for this approximation can be found in [6].

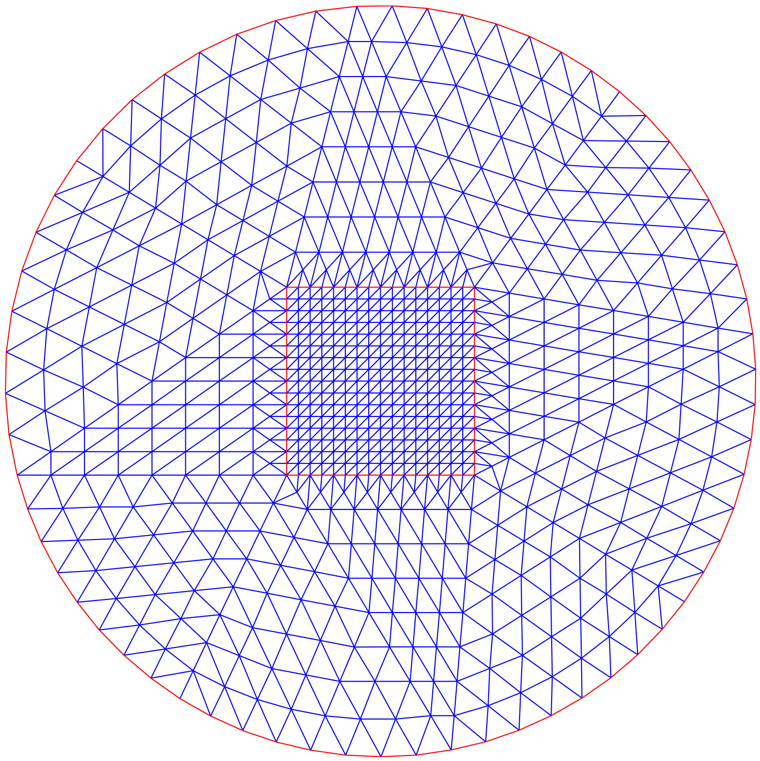

FIG. 6.1. *A typical mesh used for numerical simulations.*

6. Numerical results. In this section we present three sets of computational simulations. All results are calculated on domains of the form seen in Figure 6.1, where the superconductor is located in the square region $(-0.5, 0.5) \times (-0.5, 0.5)$. For all simulations the critical current density is taken to be $J_c = 1$ and the truncated sum for the nonlocal boundary inner product has $M = 5$.

In the first set (Figure 6.2) we take an applied magnetic field

$$\mathbf{H}^a = (0, \min\{t, H_{max}\}, 0)^T$$

for four values of H_{max}. For each value of H_{max} we display steady state solutions of the current density J_h. We see that while the applied magnetic field is increasing, the region in which the current takes critical values also increases.

In the second set of results (Figure 6.3) we apply an oscillating magnetic field of the form

$$\mathbf{H}^a = \left(0, 0.14 \sin \frac{\pi t}{2}, 0\right)^T, \tag{6.1}$$

and we display plots of the current density J_h at times $t = 1, 1.5, 2$, and 2.5.

In Table 6.1 we display the calculated error

$$\left\| \tilde{J}(\cdot, t^*) - J_{h,\Delta t}(\cdot, t^*) \right\|_{A^{-1}}$$

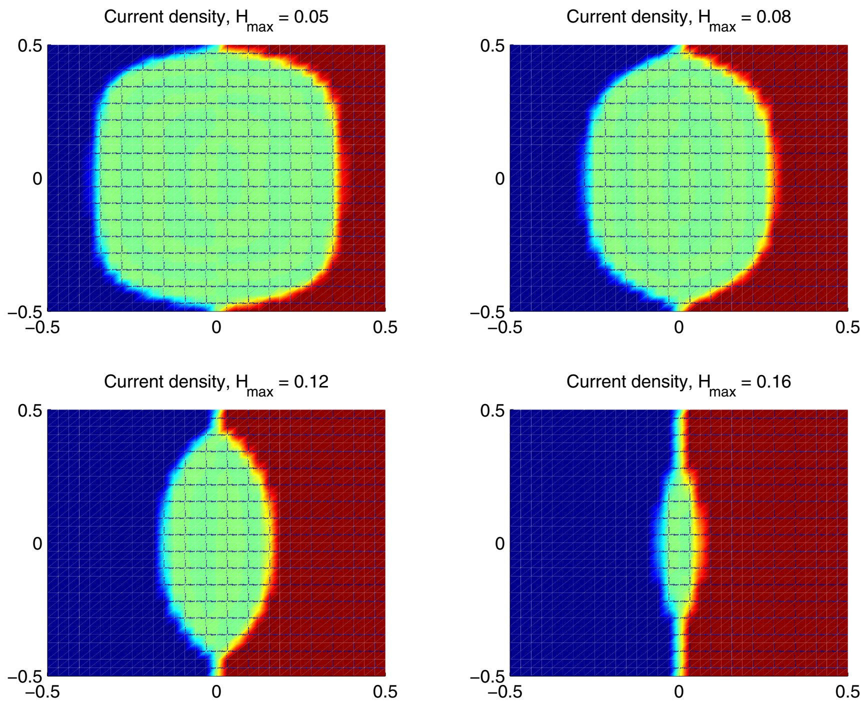

FIG. 6.2. *Steady state solutions:* $\mathbf{H}^a = (0, \min\{t, H_{max}\}, 0)^T$.

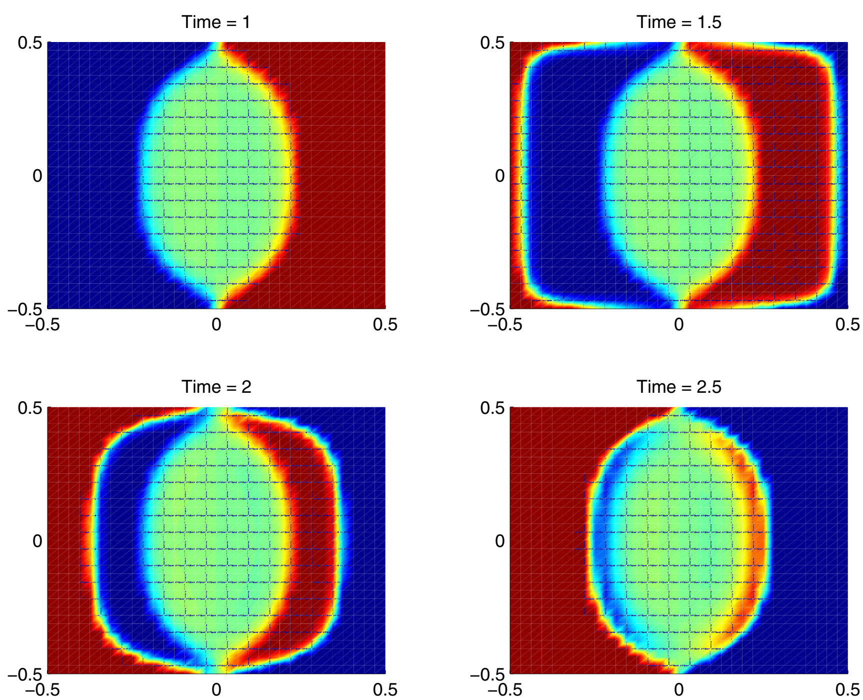

FIG. 6.3. *Current density for oscillating problem:* $\mathbf{H}^a = (0, 0.14 \sin \frac{\pi t}{2}, 0)^T$.

TABLE 6.1
Estimated errors for varying times and meshes.

	$t^* = 1.0$	$t^* = 2.0$	$t^* = 3.0$
$h = 1/8,\ \Delta t = 1/16$	0.0236	0.0255	0.0236
$h = 1/16,\ \Delta t = 1/64$	0.0126	0.0130	0.0126
$h = 1/32,\ \Delta t = 1/256$	0.0063	0.0068	0.0063
$h = 1/64,\ \Delta t = 1/1024$	0.0030	0.0037	0.0030

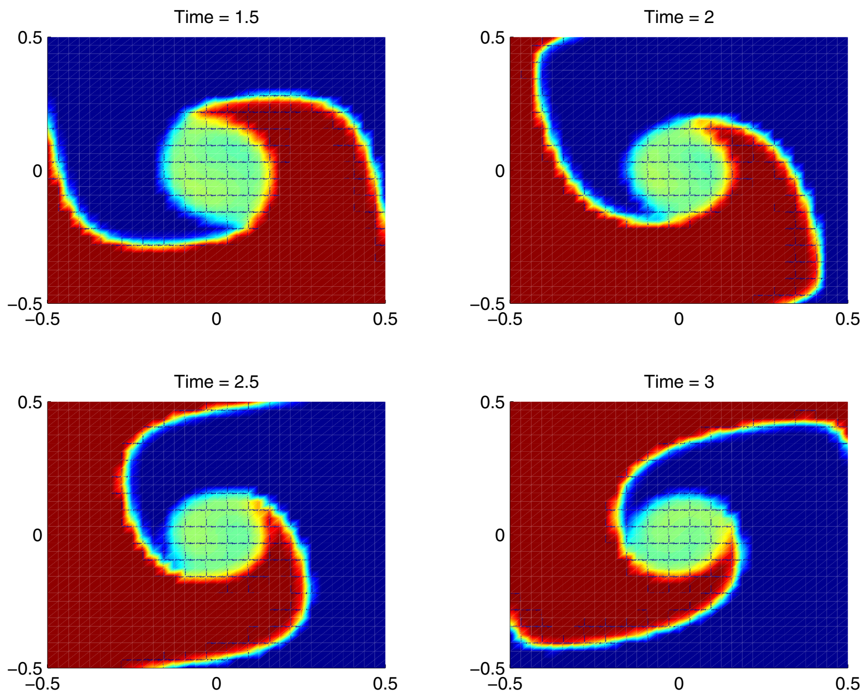

FIG. 6.4. *Current density for rotating problem:* $\mathbf{H}^a = \min\{t, 0.14\}(\sin\frac{\pi t}{2}, \cos\frac{\pi t}{2}, 0)^T$.

for the oscillating magnetic field (6.1). Here $\tilde{J}$ is the solution of $(\widehat{\mathbf{P}}_{h,\Delta t})$ obtained using a fine mesh ($h = 1/256$) and small time step ($\Delta t = 0.001$). These results are consistent with an error of $\mathcal{O}(h)$.

Finally, in Figure 6.4 we take a rotating applied magnetic field of the form

$$\mathbf{H}^a = \min\{t, 0.14\}\left(\sin\frac{\pi t}{2}, \cos\frac{\pi t}{2}, 0\right)^T,$$

and we display plots of the current density J_h at times $t = 1.5, 2, 2.5$, and 3.

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