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Numerical analysis of the TV regularization and H^{-1} fidelity model for decomposing an image into cartoon plus texture

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The Osher–Solé–Vese (OSV) model, which is the gradient flow of an energy consisting of the total variation functional plus an H^{-1} fidelity term, is studied. In this paper, we build on the analysis of the OSV model which we gave in Elliott & Smitheman (2007, *Comm. Pure Appl. Anal.*, in press). We introduce backward Euler finite-element approximations to a regularized version of the OSV initial boundary-value problem (IBVP) and to a weak formulation of the original problem. Well-posedness and unconditional Lyapunov stability of these fully discrete schemes are proved. Convergence results as the spatial mesh parameter, the time step size and the regularization parameter tend to 0 are proved. Rates of convergence as the time step size and the regularization parameter tend to 0 are found. The existence, uniqueness and Lyapunov stability of a solution to a linearly implicit finite-element approximation to the regularized version of the OSV IBVP are also proved.

Keywords: image decomposition; cartoon plus texture; TV and H^{-1} model; fourth-order parabolic equation; numerical analysis.

1. Introduction

In this paper, we consider the numerical analysis of the Osher–Solé–Vese (OSV) model (Osher *et al.*, 2003).

Let f be a greyscale image (i.e. $f\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ for some bounded open domain Ω, where f measures the greyscale intensity) which has been formed by adding Gaussian noise n of known standard deviation σ to a 'clean' image g:

$$f = g + n.$$

Clearly, without explicit knowledge of n the recovery of g from f is not possible. One approach to recovering the information in g from f is to apply a 'cartoon plus texture' model which splits f into two parts u and v:

$$f = u + v,$$

where u consists of the objects present in g (the 'cartoon' part of g) and v consists of the small-scale oscillations present in f (n plus the texture in g). The aim is to recover the 'cartoon' part u. The OSV model (Osher *et al.*, 2003) is to define u as the solution of the minimization problem

$$\inf_{u\in \mathrm{BV}(\Omega)\cap\mathcal{F}} J_\lambda(u), \qquad J_\lambda(u) := \int_\Omega |\nabla u| + \frac{\lambda}{2}\| f - u\|_{-1}^2, \tag{1.1}$$

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where $\mathrm{BV}(\Omega)$ is the space of functions of bounded variation:

$$\mathrm{BV}(\Omega) := \left\{u \in L^1(\Omega) \colon \int_\Omega |\nabla u| < \infty\right\},$$

$$\int_\Omega |\nabla u| := \sup_{v \in X} \int_\Omega u \nabla \cdot v \,\mathrm{d}\boldsymbol{x} \quad \left(= \int_\Omega |\nabla u| \mathrm{d}\boldsymbol{x} \text{ if } u \in C^1(\Omega)\right),$$

$$X := \{v = (v_1, \ldots, v_d) \in [C_0^1(\Omega)]^d \colon \|v_i\|_\infty \leqslant 1 \; \forall\, i = 1, \ldots, d\},$$

and $\mathrm{BV}(\Omega)$ is equipped with

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_1 + \int_\Omega |\nabla u|.$$

We set

$$\mathcal{F} := \{\eta \in (H^1(\Omega))' \colon \langle \eta, 1 \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ such that

$$\langle \eta, \xi \rangle = \int_\Omega \eta \xi \,\mathrm{d}\boldsymbol{x} \quad \forall\, \eta \in L^{\frac{6}{5}}(\Omega), \;\; \xi \in H^1(\Omega), \;\; d = 2, 3;$$

the right-hand side is being well-defined due to the continuous imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for $d = 2, 3$.

In $J_\lambda(u)$, the BV seminorm $\int_\Omega |\nabla u|$ is a regularizing term introduced to remove the texture, $\lambda > 0$ is a weighting parameter and the H^{-1} norm $\|f - u\|_{-1}$ is a fidelity term. The solution of the minimization problem (1.1) is the steady state of the H^{-1} gradient flow for $J_\lambda(\cdot)$, which we express as two coupled second-order equations (cf. Elliott *et al.*, 1989; Elliott & Mikelic, 1991):

$$\mathcal{G}u_t = -w - \lambda \mathcal{G}(u - f) \quad \text{in } \Omega, \tag{1.2}$$

$$w = -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{in } \Omega, \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\mathcal{G}$ is the inverse Laplacian operator under zero Neumann boundary conditions (defined formally in Section 2.2). We consider the following initial boundary-value problem (IBVP):

(P) given $T > 0$, find $u(\boldsymbol{x}, t), w(\boldsymbol{x}, t) \colon \Omega_T := \Omega \times (0, T] \to \mathbb{R}$ such that

$$u_t(\boldsymbol{x}, t) - \Delta w(\boldsymbol{x}, t) = -\lambda(u(\boldsymbol{x}, t) - f(\boldsymbol{x})) \quad \forall\, (\boldsymbol{x}, t) \in \Omega_T, \tag{1.5}$$

$$w(\boldsymbol{x}, t) = -\nabla \cdot \left(\frac{\nabla u(\boldsymbol{x}, t)}{|\nabla u(\boldsymbol{x}, t)|}\right) \quad \forall\, (\boldsymbol{x}, t) \in \Omega_T, \tag{1.6}$$

$$\frac{\partial u}{\partial \nu}(\boldsymbol{x}, t) = \frac{\partial w}{\partial \nu}(\boldsymbol{x}, t) = 0 \quad \forall\, (\boldsymbol{x}, t) \in \partial\Omega_T,$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in \Omega, \tag{1.7}$$

where $\partial\Omega_T := \partial\Omega \times (0, T]$. It was shown in Elliott & Smitheman (2007) that equations (1.2–1.4) are equivalent to equations (1.5–1.7).

Since $\frac{\nabla u}{|\nabla u|}$ is not defined when $\nabla u = 0$, a standard regularized version $J_{\lambda,\epsilon}(\cdot)$ of the energy functional $J_\lambda(\cdot)$ was introduced:

$$J_{\lambda,\epsilon}(u) := \int_\Omega |\nabla u|_\epsilon \, d\boldsymbol{x} + \frac{\lambda}{2}\| f - u\|_{-1}^2,$$

where $\epsilon > 0$ is a small regularization parameter and

$$|\boldsymbol{p}|_\epsilon := \sqrt{|\boldsymbol{p}|^2 + \epsilon^2} = \sqrt{p_1^2 + \cdots + p_n^2 + \epsilon^2}, \quad \text{for } \boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}^d.$$

The following analogue of **(P)** for $J_{\lambda,\epsilon}(\cdot)$ $(\epsilon > 0)$ was introduced in Elliott & Smitheman (2007):
(P$^\epsilon$) given $T > 0$, find $u_\epsilon(\boldsymbol{x}, t), w_\epsilon(\boldsymbol{x}, t)\colon \Omega_T \to \mathbb{R}$ such that

$$\begin{aligned}
&u_{\epsilon,t}(\boldsymbol{x}, t) - \Delta w_\epsilon(\boldsymbol{x}, t) = -\lambda(u_\epsilon(\boldsymbol{x}, t) - f(\boldsymbol{x})) && \forall\, (\boldsymbol{x}, t) \in \Omega_T,\\
&w_\epsilon(\boldsymbol{x}, t) = -\nabla \cdot \left(\frac{\nabla u_\epsilon(\boldsymbol{x}, t)}{|\nabla u_\epsilon(\boldsymbol{x}, t)|_\epsilon}\right) && \forall\, (\boldsymbol{x}, t) \in \Omega_T,\\
&u_\epsilon(\boldsymbol{x}, 0) = u_{0,\epsilon}(\boldsymbol{x}) && \forall\, \boldsymbol{x} \in \Omega,\\
&\frac{\partial u_\epsilon}{\partial \nu}(\boldsymbol{x}, t) = \frac{\partial w_\epsilon}{\partial \nu}(\boldsymbol{x}, t) = 0 && \forall\, (\boldsymbol{x}, t) \in \partial\Omega_T.
\end{aligned}$$

Since the expression $\frac{\nabla u}{|\nabla u|}$ is not defined when $\nabla u = \boldsymbol{0}$, the partial differential equations (PDEs) (1.5) and (1.6) are only formal statements. In Elliott & Smitheman (2007), we used convex analysis and variational inequalities to give a rigorous definition of a solution to **(P)**. We give the definition for **(P$^\epsilon$)** and note that a solution of **(P)** may be defined similarly (see Elliott & Smitheman, 2007).

DEFINITION 1.1 Let $\Omega \subset \mathbb{R}^d$ $(2 \leqslant d \leqslant 3)$ be a bounded open domain with Lipschitz boundary $\partial\Omega$ and suppose that $u_{0,\epsilon} \in \mathrm{BV}(\Omega) \cap \mathcal{F}$ and $f \in \mathcal{F}$. Then u_ϵ is said to be a weak solution of the IBVP **(P$^\epsilon$)** if $u_\epsilon \in C(0, T; \mathcal{F}) \cap L^\infty(0, T; \mathrm{BV}(\Omega)) \cap H^1(0, T; \mathcal{F})$, $u_\epsilon(0) = u_{0,\epsilon}$ a.e. and u_ϵ satisfies for any $s \in [0, T]$,

$$\int_0^s \langle u_\epsilon'(t), \mathcal{G}[v(t) - u_\epsilon(t)]\rangle dt + \int_0^s [J_{\lambda,\epsilon}(v(t)) - J_{\lambda,\epsilon}(u_\epsilon(t))]\, dt \geqslant 0$$
$$\forall\, v \in L^1(0, T; \mathrm{BV}(\Omega)) \cap L^2(0, T; \mathcal{F}). \tag{1.8}$$

As in Elliott & Smitheman (2007), the inequality (1.8) is equivalent to

$$\begin{aligned}
&\int_0^s \langle v'(t), \mathcal{G}[v(t) - u_\epsilon(t)]\rangle dt + \int_0^s [J_{\lambda,\epsilon}(v(t)) - J_{\lambda,\epsilon}(u_\epsilon(t))]\, dt\\
&\quad \geqslant \frac{1}{2}[\|v(s) - u_\epsilon(s)\|_{-1}^2 - \|v(0) - u_{0,\epsilon}\|_{-1}^2]\\
&\quad \forall\, v \in L^1(0, T; \mathrm{BV}(\Omega)) \cap C(0, T; \mathcal{F})\colon v' \in L^2(0, T; \mathcal{F}).
\end{aligned} \tag{1.9}$$

In Elliott & Smitheman (2007), we showed the existence of unique solutions to the minimization problems, the well-posedness of the weak formulations of **(P)** and **(P$^\epsilon$)**, the convergence of a weak solution to the minimizer of the corresponding energy and the well-posedness of the regularization procedure in the evolutionary setting (see, respectively, Theorems 3.1, 4.1, 5.1 and 6.1 therein).

This paper is organized as follows. Preliminaries for spatial and temporal discretization are given in Section 2. In Section 3, the fully discrete problems are introduced and shown to be well-posed and to have unconditionally Lyapunov stable solutions. Convergence results and rates of convergence are proved in Section 4.

2. Preliminaries

In Section 2.1, elementary algebraic inequalities for $|\cdot|_\epsilon$ are given. The inverse Neumann Laplacian operator is defined rigorously in Section 2.2. Preliminaries for spatial and temporal discretizations are given in Sections 2.3 and 2.4, respectively.

2.1 *Elementary algebraic inequalities*

LEMMA 2.1 For $\epsilon \geqslant 0$,

$$-|\boldsymbol{p}-\boldsymbol{q}| \leqslant \frac{(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{q}}{|\boldsymbol{q}|_\epsilon} \leqslant |\boldsymbol{p}|_\epsilon - |\boldsymbol{q}|_\epsilon \leqslant \frac{(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{p}}{|\boldsymbol{p}|_\epsilon} \leqslant |\boldsymbol{p}-\boldsymbol{q}| \quad \forall\, \boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^d, \tag{2.1}$$

$$\frac{(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{q}}{|\boldsymbol{p}|_\epsilon} \leqslant |\boldsymbol{p}|_\epsilon - |\boldsymbol{q}|_\epsilon \leqslant \frac{(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{p}}{|\boldsymbol{q}|_\epsilon} \quad \forall\, \boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^d. \tag{2.2}$$

Proof. The inequalities (2.1) were proved in Elliott & Smitheman (2007).

The first inequality in (2.2) follows from the second one on interchanging the roles of $\boldsymbol{p}$ and $\boldsymbol{q}$, and the second inequality can be proved as follows:

$$\begin{aligned}\forall\, \boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^d, \quad |\boldsymbol{p}|_\epsilon - |\boldsymbol{q}|_\epsilon \leqslant \frac{|\boldsymbol{p}|^2 - |\boldsymbol{p}||\boldsymbol{q}|}{|\boldsymbol{q}|_\epsilon} &\Leftrightarrow (|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2 + \epsilon^2)^2 \geqslant (|\boldsymbol{p}||\boldsymbol{q}| + |\boldsymbol{p}|_\epsilon|\boldsymbol{q}|_\epsilon)^2 \\ &\Leftrightarrow |\boldsymbol{p}|^2|\boldsymbol{p}|_\epsilon^2 + |\boldsymbol{q}|^2|\boldsymbol{q}|_\epsilon^2 \geqslant 2|\boldsymbol{p}||\boldsymbol{p}|_\epsilon|\boldsymbol{q}||\boldsymbol{q}|_\epsilon \\ &\Leftrightarrow (|\boldsymbol{p}||\boldsymbol{p}|_\epsilon - |\boldsymbol{q}||\boldsymbol{q}|_\epsilon)^2 \geqslant 0.\end{aligned}$$

□

2.2 *The inverse Neumann Laplacian operator*

We use the space

$$\mathcal{V} := \{\eta \in H^1(\Omega)\colon (\eta, 1) = 0\}.$$

By $\mathcal{G}\colon \mathcal{F} \to \mathcal{V}$ is denoted (minus) the inverse Laplacian operator under Neumann boundary conditions

$$(\nabla\mathcal{G}\eta, \nabla\xi) = \langle \eta, \xi \rangle \quad \forall\, \xi \in H^1(\Omega),$$

and $\mathcal{F}$ is equipped with the norm

$$\|\eta\|_{-1} := \|\nabla\mathcal{G}\eta\| \quad \forall\, \eta \in \mathcal{F}.$$

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and inner product on $L^2(\Omega)$. The Poincaré inequality

$$\|\eta\| \leqslant C_P(|(\eta, 1)| + \|\nabla\eta\|) \quad \forall\, \eta \in H^1(\Omega) \tag{2.3}$$

will be used. It is easy to show that

$$\|\eta\|_{-1} \leqslant C_P \|\eta\| \quad \forall\, \eta \in \mathcal{F} \cap L^2(\Omega). \tag{2.4}$$

It is easy to see that the following lemma holds.

LEMMA 2.2

(i) For $v \in \mathcal{F}$, the map $F_v \colon \mathcal{F} \to \mathbb{R}$ defined by

$$F_v(\eta) = \langle v, \mathcal{G}\eta \rangle \quad \forall\, \eta \in \mathcal{F}$$

satisfies $F_v \in \mathcal{F}'$.

(ii) For $v \in L^2(0, T; \mathcal{F})$, the map $\hat{F}_v \colon L^2(0, T; \mathcal{F}) \to \mathbb{R}$ defined by

$$\hat{F}_v(\eta) = \int_0^T \langle v(t), \mathcal{G}\eta(t) \rangle \mathrm{d}t \quad \forall\, \eta \in L^2(0, T; \mathcal{F})$$

satisfies $\hat{F}_v \in (L^2(0, T; \mathcal{F}))'$.

2.3 *The finite-element approximation*

Finite-element approximations are considered under the following assumptions on the mesh.

(A) Let $\Omega \subset \mathbb{R}^d$, $d \leqslant 3$, be a convex polygonal domain if $d = 2$ and a convex polyhedral domain if $d = 3$. Let $\{\mathcal{T}^h\}_{h>0}$ be a 'quasi-uniform' (or 'regular') family of partitioning of Ω into disjoint open simplices τ, with $h_\tau := \operatorname{diam} \tau$ and $h := \max_{\tau \in \mathcal{T}^h} h_\tau$, so that $\overline{\Omega} = \cup_{\tau \in \mathcal{T}^h} \overline{\tau}$. In addition, it is assumed that the partitioning $\mathcal{T}^h$ is

- weakly acute for $d = 2$: the sum of opposite angles relative to any side does not exceed π;
- acute for $d = 3$: the angle between any two faces of the same tetrahedron does not exceed $\frac{\pi}{2}$.

Associated with $\mathcal{T}^h$ is the standard finite-element space of continuous piecewise linear basis functions

$$S^h := \{\eta \in C(\overline{\Omega}) \colon \eta|_\tau \text{ is linear } \forall\, \tau \in \mathcal{T}^h\} \subset H^1(\Omega).$$

Let $\{\varphi_j\}_{j=1}^{N_h}$ be the canonical basis of S^h, satisfying $\varphi_j(\boldsymbol{x}_i) = \delta_{ij}$, where $\{\boldsymbol{x}_j\}_{j=1}^{N_h}$ is the set of nodes of $\mathcal{T}^h$. Take $\pi^h \colon C(\overline{\Omega}) \to S^h$ to be the interpolation operator such that $\pi^h \eta(\boldsymbol{x}_j) = \eta(\boldsymbol{x}_j)$ for $j = 1, \ldots, N_h$. Let $(\cdot, \cdot)^h$ be the following discrete inner product on $C(\overline{\Omega})$:

$$(\eta, \xi)^h := \int_\Omega \pi^h(\eta\xi) \mathrm{d}\boldsymbol{x} = \sum_{j=1}^{N_h} \hat{M}_{jj}\, \eta(\boldsymbol{x}_j) \xi(\boldsymbol{x}_j) = \boldsymbol{\eta}^T \hat{M} \boldsymbol{\xi} \quad \forall\, \eta, \xi \in C(\overline{\Omega}),$$

where $\hat{M}_{jj} := (1, \varphi_j)$ ($j = 1, \ldots, N_h$) defines the diagonal 'lumped mass' matrix and

$$[\boldsymbol{\eta}]_j := \eta(\boldsymbol{x}_j), \quad [\boldsymbol{\xi}]_j := \xi(\boldsymbol{x}_j) \quad \text{for } j = 1, \ldots, N_h.$$

The mass and stiffness matrices, M and K, respectively, are defined by

$$M_{ij} = (\varphi_i, \varphi_j), \quad K_{ij} = (\nabla\varphi_i, \nabla\varphi_j) \quad \forall\, i, j = 1, \ldots, N_h.$$

The discrete inner product $(\cdot,\cdot)^h$ induces a norm on $S^h \subset C(\overline{\Omega})$, which will be denoted by $\|\cdot\|_h$:

$$\|\eta_h\|_h := \sqrt{(\eta_h, \eta_h)^h} \quad \forall\, \eta_h \in S^h.$$

It is well known that $\|\cdot\|_h$ and $\|\cdot\|$ are equivalent on S^h:

$$c\|\eta_h\| \leqslant \|\eta_h\|_h \leqslant C\|\eta_h\| \quad \forall\, \eta_h \in S^h. \tag{2.5}$$

Since $\mathcal{T}^h$ is quasi-uniform, we have the following inverse estimate:

$$\|\nabla \eta_h\| \leqslant \frac{C}{h}\|\eta_h\|_h \quad \forall\, \eta_h \in S^h.$$

We also have the following estimate for the error due to numerical integration:

$$|(\eta_h, \xi_h) - (\eta_h, \xi_h)^h| \leqslant Ch^2\|\nabla \eta_h\|\|\nabla \xi_h\| \leqslant Ch\|\eta_h\|\|\nabla \xi_h\| \quad \forall\, \eta_h, \xi_h \in S^h. \tag{2.6}$$

We define the discrete analogues $\mathcal{G}^h\colon \mathcal{F} \to V^h$ and $\hat{\mathcal{G}}^h\colon \mathcal{F}^h \to V^h$ of the Green's operator $\mathcal{G}$ by

$$(\nabla \mathcal{G}^h \eta, \nabla \xi_h) = \langle \eta, \xi_h \rangle \quad \forall\, \xi_h \in S^h,$$

$$(\nabla \hat{\mathcal{G}}^h \eta, \nabla \xi_h) = (\eta, \xi_h)^h \quad \forall\, \xi_h \in S^h,$$

where

$$V^h := \{\eta_h \in S^h\colon (\eta_h, 1) = (\eta_h, 1)^h = 0\} \subset \mathcal{V}, \quad \mathcal{F}^h := \{\eta \in C(\overline{\Omega})\colon (\eta, 1)^h = 0\} \supset V^h.$$

We denote by $\|\cdot\|_{-h}$, $\|\cdot\|_{-h,h}$ the norms induced by $\mathcal{G}^h$, $\hat{\mathcal{G}}^h$ on $\mathcal{F}$, $\mathcal{F}^h$, respectively:

$$\|\eta\|_{-h} := \|\nabla \mathcal{G}^h \eta\| \quad \forall\, \eta \in \mathcal{F},$$

$$\|\eta\|_{-h,h} := \|\nabla \hat{\mathcal{G}}^h \eta\| \quad \forall\, \eta \in \mathcal{F}^h.$$

Unless explicitly stated otherwise, V^h and $\mathcal{F}^h$ are assumed to be equipped with $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{-h,h}$, respectively. By $(\mathcal{F}, \|\cdot\|_{-h})$ will be meant $\mathcal{F}$ equipped with $\|\cdot\|_{-h}$.

We have that

- $V^h \hookrightarrow\hookrightarrow H^1(\Omega)$ (since $V^h \hookrightarrow H^1(\Omega)$ and V^h is finite dimensional);
- V^h is a closed subspace of $H^1(\Omega)$.

We will use the following results (see Barrett & Blowey, 1995; Blowey & Elliott, 1992; Nochetto, 1991):

$$C_1 h^2\|\nabla \eta_h\| \leqslant C_2 h\|\eta_h\| \leqslant \|\eta_h\|_{-h} \leqslant \|\eta_h\|_{-1} \leqslant C_3\|\eta_h\|_{-h} \quad \forall\, \eta_h \in V^h, \tag{2.7}$$

$$h^2\|\nabla \eta_h\| \leqslant C_1 h\|\eta_h\|_h \leqslant C_2\|\eta_h\|_{-h,h} \leqslant C_3\|\eta_h\|_{-h} \leqslant C_4\|\eta_h\|_{-h,h} \quad \forall\, \eta_h \in V^h, \tag{2.8}$$

$$\|(\mathcal{G}^h - \hat{\mathcal{G}}^h)\eta_h\|_{H^1(\Omega)} \leqslant Ch^2\|\nabla \eta_h\| \leqslant Ch\|\eta_h\| \quad \forall\, \eta_h \in V^h, \tag{2.9}$$

$$\|(\mathcal{G} - \mathcal{G}^h)\eta\| \leqslant Ch\|\eta\|_{-1} \quad \forall\, \eta \in \mathcal{F}. \tag{2.10}$$

It follows from inequalities (2.4) and (2.10) that

$$\|(\mathcal{G}-\mathcal{G}^h)\eta\| \leqslant Ch\|\eta\| \quad \forall\, \eta \in \mathcal{F}\cap L^2(\Omega). \tag{2.11}$$

The triangle inequality and inequalities (2.9) and (2.11) give that

$$\|(\mathcal{G}-\hat{\mathcal{G}}^h)\eta_h\| \leqslant Ch\|\eta_h\| \quad \forall\, \eta_h \in V^h. \tag{2.12}$$

The Poincaré inequality (2.3) and the inequalities (2.5) give that for $\eta \in \mathcal{F}^h$,

$$\|\eta\|_{-h,h}^2 = (\eta, \hat{\mathcal{G}}^h\eta)^h \leqslant \|\eta\|_h\|\hat{\mathcal{G}}^h\eta\|_h \leqslant C\|\eta\|\|\nabla\hat{\mathcal{G}}^h\eta\| = C\|\eta\|\|\eta\|_{-h,h} \Rightarrow \|\eta\|_{-h,h} \leqslant C\|\eta\|. \tag{2.13}$$

It can be shown that the following lemma holds.

LEMMA 2.3 (cf. Lemma 2.2 (ii))

(i) For $v_h \in L^2(0,T;V^h)$, the map $F_{v_h}\colon L^2(0,T;V^h) \to \mathbb{R}$ defined by

$$F_{v_h}(\eta_h) = \int_0^s (\eta_h(t), \mathcal{G}^h v_h(t))\mathrm{d}t \quad \forall\, \eta_h \in L^2(0,T;V^h)$$

satisfies $F_{v_h} \in (L^2(0,T;V^h))'$.

(ii) For $v_h \in L^2(0,T;V^h)$, the map $\hat{F}_{v_h}\colon L^2(0,T;V^h) \to \mathbb{R}$ defined by

$$\hat{F}_{v_h}(\eta_h) = \int_0^s (v_h(t), \hat{\mathcal{G}}^h\eta_h(t))^h\mathrm{d}t \quad \forall\, \eta_h \in L^2(0,T;V^h)$$

satisfies $\hat{F}_{v_h} \in (L^2(0,T;V^h))'$.

The following lemma is proved in the Appendix.

LEMMA 2.4

(i) If $\{\eta_h\}_{h>0}$ is a sequence satisfying the following:

- $\eta_h \in V^h \;\; \forall\, h > 0$,
- $\|\eta_h\|$ remains bounded as $h \downarrow 0$,

then

$$\liminf_{h\downarrow 0}\|\eta_h\|_{-h,h}^2 = \liminf_{h\downarrow 0}\|\eta_h\|_{-h}^2 = \liminf_{h\downarrow 0}\|\eta_h\|_{-1}^2.$$

(ii) If $\{\eta_h\}_{h>0}$ is a sequence satisfying the following:

- $\eta_h \in L^2(0,T;V^h) \;\; \forall\, h > 0$,
- $\|\eta_h\|_{L^2(\Omega_T)}$remains bounded as$h \downarrow 0$,

then, for all $s \in [0,T]$,

$$\liminf_{h\downarrow 0}\|\eta_h\|_{L^2(0,s;\mathcal{F}^h)}^2 = \liminf_{h\downarrow 0}\|\eta_h\|_{L^2(0,s;(\mathcal{F},\|\cdot\|_{-h}))}^2 = \liminf_{h\downarrow 0}\|\eta_h\|_{L^2(0,s;\mathcal{F})}^2.$$

(iii) If $\eta \in L^2(\Omega_T) \cap L^2(0,T;\mathcal{F})$ and $\{\eta_h\}_{h>0}$ is a sequence satisfying the following:

- $\eta_h \in L^2(0,T;V^h) \ \ \forall\, h > 0$,
- $\|\eta_h\|_{L^2(\Omega_T)}$ remains bounded as $h \downarrow 0$,
- $\eta_h \to \eta$ in $L^2(0,T;\mathcal{F})$,

then

$$\lim_{h\downarrow 0} \|\nabla[\hat{\mathcal{G}}^h \eta_h - \mathcal{G}\eta]\|_{L^2(\Omega_T)} = 0.$$

(iv) If $\{\eta_h\}_{h>0}$ and $\{\xi_h\}_{h>0}$ are sequences satisfying the following:

- $\eta_h, \xi_h \in L^2(0,T;V^h) \ \ \forall\, h > 0$,
- $\|\eta_h\|_{L^2(\Omega_T)}$ and $\|\xi_h\|_{L^2(\Omega_T)}$ remain bounded as $h \downarrow 0$,

then, for all $s \in [0,T]$,

$$\liminf_{h\downarrow 0} \|\nabla[\hat{\mathcal{G}}^h \eta_h - \mathcal{G}^h \xi_h]\|^2_{L^2(0,s;L^2(\Omega))} = \liminf_{h\downarrow 0} \|\eta_h - \xi_h\|^2_{L^2(0,s;\mathcal{F})}.$$

We define the discrete L^2-projection operator $P^h \colon L^2(\Omega) \to S^h$ by

$$(P^h\eta, \xi_h) = (\eta, \xi_h) \quad \forall\, \xi_h \in S^h.$$

Clearly, $P^h\eta \in V^h$ for $\eta \in L^2(\Omega) \cap \mathcal{F}$. Also, for $m = 1, 2$ (see Barrett *et al.*, 1999),

$$\|(I - P^h)\eta\| + h\|\nabla(I - P^h)\eta\| \leqslant Ch^m |\eta|_{H^m(\Omega)} \quad \forall\, \eta \in H^m(\Omega), \tag{2.14}$$

yielding

$$\|P^h\eta\| \leqslant (Ch+1)\|\eta\|_{H^1(\Omega)}, \quad \|\nabla P^h\eta\| \leqslant (C+1)\|\eta\|_{H^1(\Omega)} \quad \forall\, \eta \in H^1(\Omega),$$
$$\forall\, \eta \in H^2(\Omega), \quad P^h\eta \to \eta \ \text{ in } H^1(\Omega) \text{ as } h \downarrow 0. \tag{2.15}$$

Further, $(I - P^h)\eta \in L^2(\Omega) \cap \mathcal{F}$ for all $\eta \in L^2(\Omega)$ and hence the inequalities (2.4) and (2.14) give that

$$\|(I - P^h)\eta\|_{-1} \leqslant Ch\|\nabla\eta\| \quad \forall\, \eta \in H^1(\Omega). \tag{2.16}$$

Inequalities (2.4) and (2.16) yield

$$\|P^h\eta\|_{-1} \leqslant (Ch + C_P)\|\eta\|_{H^1(\Omega)} \quad \forall\, \eta \in \mathcal{V}.$$

2.4 *Discretization in time*

The interval $[0,T]$ is divided into $N > 0$ intervals $[t_{n-1}, t_n]$ $(1 \leqslant n \leqslant N)$ of equal length by taking $\Delta t = \frac{T}{N}$ and $t_n = n\Delta t$ $(0 \leqslant n \leqslant N)$. The notation $U^n \approx u(t_n)$ and $d_t U^n := \frac{U^n - U^{n-1}}{\Delta t}$ $(n = 1, \ldots, N)$ is used. The piecewise linear and constant in time interpolants, U and $\hat{U}$, respectively, of a discrete in time function $\{U^n\}_{n=0}^N$ are defined by

$$\forall\, 1 \leqslant n \leqslant N, \quad U(t) := \frac{t - t_{n-1}}{\Delta t} U^n + \frac{t_n - t}{\Delta t} U^{n-1} \ \ \forall\, t \in [t_{n-1}, t_n], \quad \hat{U}(t) := U^n \ \ \forall\, t \in [t_{n-1}, t_n).$$

It follows that

$$\forall\, 1 \leqslant n \leqslant N, \quad U'(t) = d_t U^n \quad \forall\, t \in (t_{n-1}, t_n).$$

It is easy to show that

$$(d_t U^n, U^n) = \frac{1}{2} d_t[\|U^n\|^2] + \frac{\Delta t}{2}\|d_t U^n\|^2. \tag{2.17}$$

For $0 \leqslant n \leqslant N$,

$$\|U'_\epsilon\|^2_{L^2(0,t_n;\mathcal{F})} = \int_0^{t_n} \|U'_\epsilon(t)\|^2_{-1}\,\mathrm{d}t = \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|d_t U^j_\epsilon\|^2_{-1}\,\mathrm{d}t = \Delta t \sum_{j=1}^{n} \|d_t U^j_\epsilon\|^2_{-1}. \tag{2.18}$$

3. Well-posedness of fully discrete problems

In Section 3.1, we introduce the analogues of the energies $J_{\lambda,\epsilon}(\cdot)$ and $J_\lambda(\cdot)$ that we will use when considering spatially discrete problems. Fully discrete analogues of **(P)** and **(P$^\epsilon$)** are introduced in Section 3.2. Weaker formulations of these fully discrete problems are given in Section 3.3. Well-posedness and Lyapunov stability of the fully discrete solutions are given in Section 3.4.

3.1 *Energies for spatial discretization*

We consider the following analogues of the energies $J_{\lambda,\epsilon}(\cdot)$ and $J_\lambda(\cdot)$:

$$\hat{J}_{\lambda,\epsilon,h}(u) := \int_\Omega |\nabla u|_\epsilon + \frac{\lambda}{2}\|\nabla[\hat{\mathcal{G}}^h u - \mathcal{G}^h f]\|^2, \quad \hat{J}_{\lambda,h}(u) := \hat{J}_{\lambda,0,h}(u) \quad \forall\, u \in \mathrm{BV}(\Omega) \cap \mathcal{F}^h.$$

If explicit reference to the function f needs to be made, then we use the notation $J_{\lambda,\epsilon}(u, f)$, etc.

3.2 *Fully discrete problems*

We consider the linearly and fully implicit approximations to the fourth-order PDE (1.2):

($\mathbf{P}_1^{\epsilon,h,\Delta t}$) find $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ such that $U^0_{\epsilon,h} = u_{0,\epsilon,h} \in V^h$ and for $n = 1, \ldots, N$,

$$(\hat{\mathcal{G}}^h d_t U^n_{\epsilon,h}, \eta_h)^h + \left(\frac{\nabla U^n_{\epsilon,h}}{|\nabla U^{n-1}_{\epsilon,h}|_\epsilon}, \nabla \eta_h\right) = -\lambda(\hat{\mathcal{G}}^h U^n_{\epsilon,h} - \mathcal{G}^h f, \eta_h)^h \quad \forall\, \eta_h \in S^h; \tag{3.1}$$

($\mathbf{P}_2^{\epsilon,h,\Delta t}$) find $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ such that $U^0_{\epsilon,h} = u_{0,\epsilon,h} \in V^h$ and for $n = 1, \ldots, N$,

$$(\hat{\mathcal{G}}^h d_t U^n_{\epsilon,h}, \eta_h)^h + \left(\frac{\nabla U^n_{\epsilon,h}}{|\nabla U^n_{\epsilon,h}|_\epsilon}, \nabla \eta_h\right) = -\lambda(\hat{\mathcal{G}}^h U^n_{\epsilon,h} - \mathcal{G}^h f, \eta_h)^h \quad \forall\, \eta_h \in S^h. \tag{3.2}$$

We also consider the following fully implicit approximation to the unregularized analogue of the variational inequality (1.8):

($\mathbf{P}^{h,\Delta t}$) find $\{U^n_h\}_{n=1}^N \subset V^h$ such that $U^0_h = u_{0,h} \in V^h$ and for $n = 1, \ldots, N$,

$$(d_t U^n_h, \hat{\mathcal{G}}^h[v_h - U^n_h])^h + \hat{J}_{\lambda,h}(v_h) - \hat{J}_{\lambda,h}(U^n_h) \geqslant 0 \quad \forall\, v_h \in V^h. \tag{3.3}$$

3.3 *Preliminary results*

In this section, we give variational inequalities that follow from the equation (3.2) for $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ and the inequality (3.3) for $(\mathbf{P}^{h,\Delta t})$. Lemma 3.1 gives that the natural analogue of the variational inequality (3.3) for $(\mathbf{P}^{h,\Delta t})$ can be deduced from the equation (3.2) for $(\mathbf{P}_2^{\epsilon,h,\Delta t})$. Lemma 3.2 gives analogues of the variational inequality (1.9) for $(\mathbf{P}^{\epsilon})$ which can be deduced from the equation (3.2) for $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ and the inequality (3.3) for $(\mathbf{P}^{h,\Delta t})$. Lemma 3.3 gives equivalent formulations of the inequality (3.3) for $(\mathbf{P}^{h,\Delta t})$ and the analogous inequality for $(\mathbf{P}_2^{\epsilon,h,\Delta t})$.

LEMMA 3.1 If $\{U_{\epsilon,h}^n\}_{n=1}^N \subset V^h$ is a solution of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$, then

$$\forall n = 1, \ldots, N, \quad (d_t U_{\epsilon,h}^n, \hat{\mathcal{G}}^h[v_h - U_{\epsilon,h}^n])^h + \hat{J}_{\lambda,\epsilon,h}(v_h) - \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}^n) \geqslant 0 \quad \forall v_h \in V^h. \tag{3.4}$$

Proof. Suppose that $\{U_{\epsilon,h}^n\}_{n=1}^N \subset V^h$ is a solution of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$. Take $v_h \in V^h$. Letting $\eta_h = v_h - U_{\epsilon,h}^n$ in equation (3.2) and using the inequality $(a-b)(c-a) \leqslant \frac{1}{2}[(c-b)^2 - (a-b)^2]$ and inequality (2.1) give inequality (3.4). □

LEMMA 3.2 If $\{U_h^n\}_{n=0}^N, \{U_{\epsilon,h}^n\}_{n=0}^N \subset V^h$ are solutions of $(\mathbf{P}^{h,\Delta t})$, $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ and $U_h, U_{\epsilon,h}$ are their piecewise linear in time interpolants, then, for all $s \in [0, T]$,

$$\int_0^s (v_h'(t), \hat{\mathcal{G}}^h[v_h(t) - U_h(t)])^h \,\mathrm{d}t + \int_0^s [\hat{J}_{\lambda,h}(v_h(t)) - \hat{J}_{\lambda,h}(\hat{U}_h(t))]\mathrm{d}t$$
$$\geqslant \frac{1}{2}[\|v_h(s) - U_h(s)\|_{-h,h}^2 - \|v_h(0) - u_{0,h}\|_{-h,h}^2] \quad \forall v_h \in H^1(0,T;V^h) \cap C(0,T;V^h); \tag{3.5}$$

$$\int_0^s (v_h'(t), \hat{\mathcal{G}}^h[v_h(t) - U_{\epsilon,h}(t)])^h \mathrm{d}t + \int_0^s [\hat{J}_{\lambda,\epsilon,h}(v_h(t)) - \hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t))]\mathrm{d}t$$
$$\geqslant \frac{1}{2}[\|v_h(s) - U_{\epsilon,h}(s)\|_{-h,h}^2 - \|v_h(0) - u_{0,\epsilon,h}\|_{-h,h}^2] \quad \forall v_h \in H^1(0,T;V^h) \cap C(0,T;V^h). \tag{3.6}$$

Proof. We prove the result for $(\mathbf{P}_2^{\epsilon,h,\Delta t})$; the result for $(\mathbf{P}^{h,\Delta t})$ can be proved analogously. Indeed, the inequality (3.4) implies that, for a.e. $t \in [0, T]$,

$$(U_{\epsilon,h}'(t), \hat{\mathcal{G}}^h[v_h - \hat{U}_{\epsilon,h}(t)])^h + \hat{J}_{\lambda,\epsilon,h}(v_h) - \hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t)) \geqslant 0 \quad \forall v_h \in V^h.$$

It follows that for all $s \in [0, T]$,

$$\int_0^s (U_{\epsilon,h}'(t), \hat{\mathcal{G}}^h[v_h(t) - U_{\epsilon,h}(t)])^h \mathrm{d}t + \int_0^s (U_{\epsilon,h}'(t), \hat{\mathcal{G}}^h[U_{\epsilon,h}(t) - \hat{U}_{\epsilon,h}(t)])^h \,\mathrm{d}t$$
$$+ \int_0^s [\hat{J}_{\lambda,\epsilon,h}(v_h(t)) - \hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t))]\mathrm{d}t \geqslant 0 \quad \forall v_h \in C(0,T;V^h).$$

Further,

$$(v_h'(t) - U_{\epsilon,h}'(t), \hat{\mathcal{G}}^h[v_h(t) - U_{\epsilon,h}(t)])^h = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|v_h(t) - U_{\epsilon,h}(t)\|_{-h,h}^2].$$

Hence, for all $s \in [0, T]$,

$$\int_0^s (v_h'(t), \hat{\mathcal{G}}^h[v_h(t) - U_{\epsilon,h}(t)])^h \mathrm{d}t$$
$$+ \int_0^s (U_{\epsilon,h}'(t), \hat{\mathcal{G}}^h[U_{\epsilon,h}(t) - \hat{U}_{\epsilon,h}(t)])^h \,\mathrm{d}t + \int_0^s [\hat{J}_{\lambda,\epsilon,h}(v_h(t)) - \hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t))] \,\mathrm{d}t$$
$$\geqslant \frac{1}{2}[\|v_h(s) - U_{\epsilon,h}(s)\|_{-h,h}^2 - \|v_h(0) - u_{0,\epsilon,h}\|_{-h,h}^2] \quad \forall\, v_h \in H^1(0,T;V^h) \cap C(0,T;V^h).$$

Take $s \in [0, T]$ and $n \in [1, N]$ such that $s \in [t_{n-1}, t_n]$. Then

$$\int_0^s (U_{\epsilon,h}'(t), \hat{\mathcal{G}}^h[U_{\epsilon,h}(t) - \hat{U}_{\epsilon,h}(t)])^h \,\mathrm{d}t$$
$$= \sum_{j=1}^{n-1} (d_t U_{\epsilon,h}^j, \hat{\mathcal{G}}^h(d_t U_{\epsilon,h}^j))^h \int_{t_{j-1}}^{t_j} (t - t_j)\mathrm{d}t + (d_t U_{\epsilon,h}^n, \hat{\mathcal{G}}^h(d_t U_{\epsilon,h}^n))^h \int_{t_{n-1}}^{s} (t - t_n) \,\mathrm{d}t \leqslant 0. \qquad \square$$

LEMMA 3.3 For $\{U_h^n\}_{n=1}^N, \{U_{\epsilon,h}^n\}_{n=1}^N \subset V^h$, the inequalities (3.3), (3.4) are equivalent to, respectively,

$$\Delta t \sum_{j=1}^n (d_t v_h^j, \hat{\mathcal{G}}^h[v_h^j - U_h^j])^h + \Delta t \sum_{j=1}^n [\hat{J}_{\lambda,h}(v_h^j) - \hat{J}_{\lambda,h}(U_h^j)]$$
$$\geqslant \frac{1}{2}[\|U_h^n - v_h^n\|_{-h,h}^2 - \|u_{0,h} - v_h^0\|_{-h,h}^2] + \frac{\Delta t}{2}\|U_h' - v_h'\|_{L^2(0,t_n;\mathcal{F}^h)}^2 \quad \forall \{v_h^j\}_{j=0}^N \subset V^h, \tag{3.7}$$

$$\Delta t \sum_{j=1}^n (d_t v_h^j, \hat{\mathcal{G}}^h[v_h^j - U_{\epsilon,h}^j])^h + \Delta t \sum_{j=1}^n [\hat{J}_{\lambda,\epsilon,h}(v_h^j) - \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}^j)]$$
$$\geqslant \frac{1}{2}[\|U_{\epsilon,h}^n - v_h^n\|_{-h,h}^2 - \|u_{0,\epsilon,h} - v_h^0\|_{-h,h}^2] + \frac{\Delta t}{2}\|U_{\epsilon,h}' - v_h'\|_{L^2(0,t_n;\mathcal{F}^h)}^2 \quad \forall \{v_h^j\}_{j=0}^N \subset V^h, \tag{3.8}$$

where U_h, $U_{\epsilon,h}$, v_h are the piecewise linear in time interpolants of $\{U_h^j\}_{j=0}^N$, $\{U_{\epsilon,h}^j\}_{j=0}^N$, $\{v_h^j\}_{j=0}^N$, respectively.

Proof. We prove that (3.4) is equivalent to (3.8); that (3.3) is equivalent to (3.7) can be proved analogously.

To show that inequality (3.4) implies inequality (3.8), consider $\{v_h^j\}_{j=0}^N \subset V^h$. Equation (2.17) gives that

$$(d_t U_{\epsilon,h}^n, \hat{\mathcal{G}}^h[v_h^n - U_{\epsilon,h}^n])^h$$
$$= (d_t v_h^n, \hat{\mathcal{G}}^h[v_h^n - U_{\epsilon,h}^n])^h - \tfrac{1}{2} d_t \|U_{\epsilon,h}^n - v_h^n\|_{-h,h}^2 - \tfrac{\Delta t}{2}\|d_t(U_{\epsilon,h}^n - v_h^n)\|_{-h,h}^2, \quad 1 \leqslant n \leqslant N.$$

Replacing v_h by v_h^n in (3.4) and using the last equation gives that

$$\frac{1}{2} d_t \|U_{\epsilon,h}^n - v_h^n\|_{-h,h}^2 + \frac{\Delta t}{2}\|d_t(U_{\epsilon,h}^n - v_h^n)\|_{-h,h}^2 + \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}^n)$$
$$\leqslant (d_t v_h^n, \hat{\mathcal{G}}^h[v_h^n - U_{\epsilon,h}^n])^h + \hat{J}_{\lambda,\epsilon,h}(v_h^n), \quad 1 \leqslant n \leqslant N.$$

Summation gives that for $0 \leqslant n \leqslant N$,

$$\frac{1}{2\Delta t}\|U_{\epsilon,h}^n - v_h^n\|_{-h,h}^2 + \frac{\Delta t}{2}\sum_{j=1}^n \|d_t(U_{\epsilon,h}^j - v_h^j)\|_{-h,h}^2 + \sum_{j=1}^n \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}^j)$$
$$\leqslant \frac{1}{2\Delta t}\|U_{\epsilon,h}^0 - v_h^0\|_{-h,h}^2 + \sum_{j=1}^n (d_t v_h^j, \hat{\mathcal{G}}^h[v_h^j - U_{\epsilon,h}^j])^h + \sum_{j=1}^n \hat{J}_{\lambda,\epsilon,h}(v_h^j) \quad \forall \{v_h^j\}_{j=0}^N \subset V^h. \tag{3.9}$$

Equation (2.18) shows that inequality (3.9) is equivalent to inequality (3.8).

To show that inequality (3.9) gives inequality (3.4), take $v_h \in V^h$ and $1 \leqslant n \leqslant N$. Letting

$$v_h^j = U_{\epsilon,h}^j \in V^h \text{ for } 0 \leqslant j \leqslant n-1 \quad \text{and} \quad v_h^j = v_h \text{ for } n \leqslant j \leqslant N$$

in inequality (3.9) gives that

$$\frac{1}{\Delta t}\|U_{\epsilon,h}^n - v_h\|_{-h,h}^2 + \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}^n) \leqslant \frac{1}{\Delta t}(v_h - U_{\epsilon,h}^{n-1}, \hat{\mathcal{G}}^h[v_h - U_{\epsilon,h}^n])^h + \hat{J}_{\lambda,\epsilon,h}(v_h),$$

and rearranging this last inequality gives inequality (3.4). □

3.4 *Well-posedness and Lyapunov stability of fully discrete solutions*

In this section, we prove well-posedness and unconditional Lyapunov stability of solutions to $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ and $(\mathbf{P}^{h,\Delta t})$. Furthermore, we prove that there exists a unique solution to $(\mathbf{P}_1^{\epsilon,h,\Delta t})$ and that this solution is unconditionally Lyapunov stable.

LEMMA 3.4

(i) Each of the problems $(\mathbf{P}_1^{\epsilon,h,\Delta t})$ and $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ has a unique solution $\{U_{\epsilon,h}^n\}_{n=1}^N$. Moreover, in each case the piecewise linear in time interpolant $U_{\epsilon,h}$ of the solution $\{U_{\epsilon,h}^n\}_{n=1}^N$ satisfies the stability estimate

$$\left(1 + \frac{\lambda \Delta t}{2}\right)\|U'_{\epsilon,h}\|_{L^2(0,T;\mathcal{F}^h)}^2 + \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(t)) \leqslant \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) \quad \forall t \in [0,T]. \tag{3.10}$$

Let $\{U^n_{\epsilon,h,i}\}_{n=1}^N$ $(i=1,2)$ be solutions of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ for data $u_{0,\epsilon,h,i} \in V^h$, $f_i \in \mathcal{F}$. Then

$$\Big(\|U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}\|^2_{-h,h} + \Delta t\|U'_{\epsilon,h,2} - U'_{\epsilon,h,1}\|^2_{L^2(0,t_n;\mathcal{F}^h)}\Big)^{\frac{1}{2}}$$
$$\leqslant \|u_{0,\epsilon,h,2} - u_{0,\epsilon,h,1}\|_{-h,h} + \sqrt{\lambda T}\|f_2 - f_1\|_{-h} \quad \forall n = 0,\dots,N. \tag{3.11}$$

(ii) The problem $(\mathbf{P}^{h,\Delta t})$ has a unique solution $\{U^n_h\}_{n=1}^N$. Moreover, the piecewise linear in time interpolant U_h of $\{U^n_h\}_{n=1}^N$ satisfies the stability estimate

$$\|U'_h\|^2_{L^2(0,T;\mathcal{F}^h)} + \hat{J}_{\lambda,h}(U_h(t)) \leqslant \hat{J}_{\lambda,h}(u_{0,h}) \quad \forall t \in [0,T]. \tag{3.12}$$

Let $\{U^n_{h,i}\}_{n=1}^N$ $(i=1,2)$ be solutions of $(\mathbf{P}^{h,\Delta t})$ for data $u_{0,h,i} \in V^h$, $f_i \in \mathcal{F}$. Then

$$\Big(\|U^n_{h,2} - U^n_{h,1}\|^2_{-h,h} + \Delta t\|U'_{h,2} - U'_{h,1}\|^2_{L^2(0,t_n;\mathcal{F}^h)}\Big)^{\frac{1}{2}}$$
$$\leqslant \|u_{0,h,2} - u_{0,h,1}\|_{-h,h} + \sqrt{\lambda T}\|f_2 - f_1\|_{-h} \quad \forall n = 0,\dots,N. \tag{3.13}$$

Proof. We prove that

(a) there exists a solution $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$;
(b) if $\{U^n_{\epsilon,h,i}\}_{n=1}^N \subset V^h$ $(i=1,2)$ satisfy (3.4) for data $u_{0,\epsilon,h,i} \in V^h$, $f_i \in \mathcal{F}$, then they satisfy inequality (3.11);
(c) the solution $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ satisfies inequality (3.10);
(d) there exists a unique solution $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ of $(\mathbf{P}_1^{\epsilon,h,\Delta t})$;
(e) the solution $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ of $(\mathbf{P}_1^{\epsilon,h,\Delta t})$ satisfies inequality (3.10);
(f) there exists a solution $\{U^n_h\}_{n=1}^N \subset V^h$ of $(\mathbf{P}^{h,\Delta t})$;
(g) if $\{U^n_{h,i}\}_{n=1}^N$ $(i=1,2)$ are solutions of $(\mathbf{P}^{h,\Delta t})$ for data $u_{0,h,i} \in V^h$, $f_i \in \mathcal{F}$, then they satisfy inequality (3.13);
(h) the solution $\{U^n_h\}_{n=1}^N \subset V^h$ of $(\mathbf{P}^{h,\Delta t})$ satisfies inequality (3.12)

as follows.

(a) We consider the following minimization problem:

given $u_{\epsilon,h} \in V^h$, find $\overline{\xi}_h = \overline{\xi}_h(u_{\epsilon,h}) \in V^h$ such that

$$\tilde{J}_{\lambda,\epsilon,h}(\overline{\xi}_h; u_{\epsilon,h}) = \inf_{\xi_h \in V^h} \tilde{J}_{\lambda,\epsilon,h}(\xi_h; u_{\epsilon,h}),$$

where

$$\tilde{J}_{\lambda,\epsilon,h}(\xi_h; u_{\epsilon,h}) := \hat{J}_{\lambda,\epsilon,h}(\xi_h) + \frac{1}{2\Delta t}\|\xi_h - u_{\epsilon,h}\|^2_{-h,h}.$$

LEMMA 3.5 The above minimization problem has a unique solution.

Proof. Let $\{\xi_h^m\}_{m\in\mathbb{N}} \subset V^h$ be a minimizing sequence for $\tilde{J}_{\lambda,\epsilon,h}(\cdot; u_{\epsilon,h})$. There exists a constant $M > 0$ such that

$$\tilde{J}_{\lambda,\epsilon,h}(\xi_h^m; u_{\epsilon,h}) \leqslant M \quad \forall\, m \in \mathbb{N}.$$

The inequalities (2.7) yield

$$\|\xi_h^m\|_{-h,h}^2 \leqslant 2\|\nabla[\hat{\mathcal{G}}^h \xi_h^m - \mathcal{G}^h f]\|^2 + 2\|f\|_{-1}^2 \leqslant \frac{4M}{\lambda} + 2\|f\|_{-1}^2 \quad \forall\, m \in \mathbb{N}.$$

Hence, the inequalities (2.8) show that there exists $C > 0$ such that

$$\|\xi_h^m\|_{H^1(\Omega)}^2 \leqslant \frac{C}{h^4}\left[\frac{4M}{\lambda} + 2\|f\|_{-1}^2\right] \quad \forall\, m \in \mathbb{N}.$$

It follows that $\{\xi_h^m\}_{m\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Hence, there exists $\overline{\xi}_h \in V^h$ such that

$$\xi_h^m \to \overline{\xi}_h \text{ in } V^h \quad \text{as } m \to \infty.$$

Inequality (2.13) yields

$$\|\xi_h^m - u_{\epsilon,h}\|_{-h,h} \to \|\overline{\xi}_h - u_{\epsilon,h}\|_{-h,h} \quad \text{and} \quad \|\nabla[\hat{\mathcal{G}}^h \xi_h^m - \mathcal{G}^h f]\| \to \|\nabla[\hat{\mathcal{G}}^h \overline{\xi}_h - \mathcal{G}^h f]\| \quad \text{as } m \to \infty.$$

Inequality (2.1) gives

$$|(|\nabla \xi_h^m|_\epsilon, 1) - (|\nabla \overline{\xi}_h|_\epsilon, 1)| \leqslant (|\nabla(\xi_h^m - \overline{\xi}_h)|, 1) \leqslant |\Omega|^{\frac{1}{2}} \|\xi_h^m - \overline{\xi}_h\|_{H^1(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

Hence,

$$\tilde{J}_{\lambda,\epsilon,h}(\overline{\xi}_h; u_{\epsilon,h}) = \lim_{m\to\infty} \tilde{J}_{\lambda,\epsilon,h}(\xi_h^m; u_{\epsilon,h}),$$

and thus $\overline{\xi}_h$ is a solution of the minimization problem.

As in the proof of Theorem 3.1 in Elliott & Smitheman (2007), $\tilde{J}_{\lambda,\epsilon,h}(\cdot; u_{\epsilon,h})$ is strictly convex, showing that the solution of the minimization problem is unique. □

By the calculus of variations, the minimizer $\overline{\xi}_h \in V^h$ of $\tilde{J}_{\lambda,\epsilon,h}(\cdot; u_{\epsilon,h})$ satisfies

$$\left[\frac{\delta}{\delta\alpha} \tilde{J}_{\lambda,\epsilon,h}(\overline{\xi}_h + \alpha\eta_h; u_{\epsilon,h})\right]_{\alpha=0} = 0 \quad \forall\, \eta_h \in V^h,$$

yielding

$$\left(\frac{\nabla\overline{\xi}_h}{|\nabla\overline{\xi}_h|_\epsilon}, \nabla\eta_h\right) + \lambda(\hat{\mathcal{G}}^h\overline{\xi}_h - \mathcal{G}^h f, \eta_h)^h + \frac{1}{\Delta t}(\hat{\mathcal{G}}^h[\overline{\xi}_h - u_{\epsilon,h}], \eta_h)^h = 0 \quad \forall\, \eta_h \in S^h.$$

Taking $U_{\epsilon,h}^n = \overline{\xi}_h(U_{\epsilon,h}^{n-1})$ for each $n = 1, \ldots, N$ in turn gives the existence of a solution $\{U_{\epsilon,h}^n\}_{n=1}^N \subset V^h$ of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$.

(b) Let $\{U^n_{\epsilon,h,i}\}_{n=1}^N \subset V^h$ $(i = 1, 2)$ satisfy (3.4) for data $u_{0,\epsilon,h,i} \in V^h$, $f_i \in \mathcal{F}$, and define the discrete in space and time function $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ by

$$U^n_{\epsilon,h} := \frac{U^n_{\epsilon,h,1} + U^n_{\epsilon,h,2}}{2} \in V^h \quad \forall n = 0, \ldots, N \quad \left(\Rightarrow U^0_{\epsilon,h} = \frac{u_{0,\epsilon,h,1} + u_{0,\epsilon,h,2}}{2} \in V^h\right).$$

Taking $v_h = U^n_{\epsilon,h}$ in the inequalities (3.4) for $i = 1, 2$ and adding the two resulting inequalities gives

$$\begin{aligned} &-\frac{1}{2}(d_t[U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}], \hat{\mathcal{G}}^h[U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}])^h \\ &+ [2\hat{J}_{0,\epsilon,h}(U^n_{\epsilon,h}) - \hat{J}_{0,\epsilon,h}(U^n_{\epsilon,h,1}) - \hat{J}_{0,\epsilon,h}(U^n_{\epsilon,h,2})] \\ &+ \frac{\lambda}{2}[\|\nabla[\hat{\mathcal{G}}^h U^n_{\epsilon,h} - \mathcal{G}^h f_1]\|^2 + \|\nabla[\hat{\mathcal{G}}^h U^n_{\epsilon,h} - \mathcal{G}^h f_2]\|^2 \\ &- \|\nabla[\hat{\mathcal{G}}^h U^n_{\epsilon,h,1} - \mathcal{G}^h f_1]\|^2 - \|\nabla[\hat{\mathcal{G}}^h U^n_{\epsilon,h,2} - \mathcal{G}^h f_2]\|^2] \geqslant 0 \quad \forall n = 1, \ldots, N. \end{aligned}$$

Using the inequality

$$\left(\frac{a+b}{2} - c\right)^2 + \left(\frac{a+b}{2} - d\right)^2 - (a-c)^2 - (b-d)^2 \leqslant \frac{1}{2}(d-c)^2 \quad \forall a, b, c, d \in \mathbb{R},$$

equation (2.17) and the convexity of $\hat{J}_{0,\epsilon,h}$ (Acar & Vogel, 1994, Theorem 2.4) yields

$$d_t[\|U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}\|^2_{-h,h}] + \Delta t\|d_t[U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}]\|^2_{-h,h} \leqslant \lambda\|f_2 - f_1\|^2_{-h} \quad \forall n = 1, \ldots, N,$$

and summation gives that

$$\begin{aligned} &\|U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}\|^2_{-h,h} + (\Delta t)^2 \sum_{j=1}^n \|d_t[U^j_{\epsilon,h,2} - U^j_{\epsilon,h,1}]\|^2_{-h,h} \\ &\quad \leqslant \|u_{0,\epsilon,h,2} - u_{0,\epsilon,h,1}\|^2_{-h,h} + n\lambda\Delta t\|f_2 - f_1\|^2_{-h} \quad \forall n = 0, \ldots, N. \end{aligned}$$

Equation (2.18) gives

$$\begin{aligned} &\|U^n_{\epsilon,h,2} - U^n_{\epsilon,h,1}\|^2_{-h,h} + \Delta t\|U'_{\epsilon,h,2} - U'_{\epsilon,h,1}\|^2_{L^2(0,t_n;\mathcal{F}^h)} \\ &\quad \leqslant \|u_{0,\epsilon,h,2} - u_{0,\epsilon,h,1}\|^2_{-h,h} + \lambda T\|f_2 - f_1\|^2_{-h} \quad \forall n = 0, \ldots, N. \end{aligned}$$

Taking the square roots and using the fact that $\sqrt{a^2 + b^2} \leqslant |a| + |b|$ gives inequality (3.11).

(c) Take $\{U^n_{\epsilon,h}\}_{n=1}^N \subset V^h$ to be the solution of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$.

Letting $\eta_h = d_t U^n_{\epsilon,h}$ in equation (3.2) gives that

$$\Delta t\|d_t U^n_{\epsilon,h}\|^2_{-h,h} + \left(\frac{\nabla U^n_{\epsilon,h}}{|\nabla U^n_{\epsilon,h}|_\epsilon}, \nabla[U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}]\right) = -\lambda(\hat{\mathcal{G}}^h U^n_{\epsilon,h} - \mathcal{G}^h f, U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h})^h$$
$$\forall n = 1, \ldots, N.$$

Inequality (2.1) and equation (2.17) give

$$\Delta t\left(1+\frac{\lambda\Delta t}{2}\right)\|d_t U^n_{\epsilon,h}\|^2_{-h,h}+\int_\Omega|\nabla U^n_{\epsilon,h}|_\epsilon+\frac{\lambda\Delta t}{2}d_t\|\nabla[\hat{\mathcal{G}}^h U^n_{\epsilon,h}-\mathcal{G}^h f]\|^2\leqslant\int_\Omega|\nabla U^{n-1}_{\epsilon,h}|_\epsilon \quad \forall n=1,\ldots,N.$$

Summing and using equation (2.18) show that inequality (3.10) holds for $t=t_n$ $(0\leqslant n\leqslant N)$:

$$\left(1+\frac{\lambda\Delta t}{2}\right)\|U'_{\epsilon,h}\|^2_{L^2(0,t_n;\mathcal{F}^h)}+\hat{J}_{\lambda,\epsilon,h}(U^n_{\epsilon,h})\leqslant\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h})\quad\forall n=0,\ldots,N.$$

To show that inequality (3.10) holds for all $t\in[0,T]$, suppose that $t\in[t_{n-1},t_n]$ for some $1\leqslant n\leqslant N$. There exists $\alpha\in[0,1]$ such that $t=\alpha t_{n-1}+(1-\alpha)t_n$. The convexity of $\hat{J}_{\lambda,\epsilon,h}(\cdot)$ (see the proof of Theorem 3.1 in Elliott & Smitheman, 2007) gives

$$\hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(t))=\hat{J}_{\lambda,\epsilon,h}((1-\alpha)U^n_{\epsilon,h}+\alpha U^{n-1}_{\epsilon,h})\leqslant(1-\alpha)\hat{J}_{\lambda,\epsilon,h}(U^n_{\epsilon,h})+\alpha\hat{J}_{\lambda,\epsilon,h}(U^{n-1}_{\epsilon,h}).$$

Multiplying the inequalities (3.10) for $t=t_{n-1},t_n$ by α, $1-\alpha$, respectively, and adding the two resulting inequalities gives

$$\left(1+\frac{\lambda\Delta t}{2}\right)\left[(1-\alpha)\|U'_{\epsilon,h}\|^2_{L^2(0,t_n;\mathcal{F}^h)}+\alpha\|U'_{\epsilon,h}\|^2_{L^2(0,t_{n-1};\mathcal{F}^h)}\right]+\hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(t))\leqslant\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}).$$

Using

$$(1-\alpha)\|U'_{\epsilon,h}\|^2_{L^2(t_{n-1},t_n;\mathcal{F}^h)}=\|U'_{\epsilon,h}\|^2_{L^2(t_{n-1},t;\mathcal{F}^h)}$$

gives inequality (3.10).

(d) We define vectors $\boldsymbol{U}^n_{\epsilon,h}$ $(n=0,\ldots,N)$, $\boldsymbol{W}^n_{\epsilon,h}$ $(n=1,\ldots,N)$ and $\boldsymbol{f}_h$ of length N_h by

$$[\boldsymbol{U}^n_{\epsilon,h}]_i=U^n_{\epsilon,h}(\boldsymbol{x}_i),\quad[\boldsymbol{W}^n_{\epsilon,h}]_i=W^n_{\epsilon,h}(\boldsymbol{x}_i),\quad[\boldsymbol{f}_h]_i=\langle f,\varphi_i\rangle\quad\forall i=1,\ldots,N_h,$$

where $\{\boldsymbol{x}_i\}_{i=1}^{N_h}$ are the nodes of the triangulation. For a finite-element function $\eta_h\in S^h$, we define the symmetric positive semidefinite matrix $A_{\epsilon,h}(\eta_h)\in\mathbb{R}^{N_h\times N_h}$ by

$$[A_{\epsilon,h}(\eta_h)]_{i,j}=\left(\frac{\nabla\varphi_i}{|\nabla\eta_h|_\epsilon},\nabla\varphi_j\right)\quad\forall i,j\leqslant N_h.$$

It is easy to show that the fourth-order equation (3.1) is equivalent to

$$(d_tU^n_{\epsilon,h},\eta_h)^h+(\nabla W^n_{\epsilon,h},\nabla\eta_h)=-\lambda(U^n_{\epsilon,h},\eta_h)^h+\lambda\langle f,\eta_h\rangle\quad\forall\eta_h\in S^h,\tag{3.14}$$

$$(W^n_{\epsilon,h},\eta_h)^h=\left(\frac{\nabla U^n_{\epsilon,h}}{|\nabla U^{n-1}_{\epsilon,h}|_\epsilon},\nabla\eta_h\right)\quad\forall\eta_h\in S^h\tag{3.15}$$

(cf. the proof of the equivalence of (1.2), (1.3) to (1.5), (1.6) given in Elliott & Smitheman, 2007). Furthermore, the pair of second-order equations (3.14) and (3.15) is equivalent to

$$\begin{pmatrix}(1+\lambda\Delta t)\hat{M} & \Delta t K\\ (1+\lambda\Delta t)A_{\epsilon,h}(U^{n-1}_{\epsilon,h}) & -(1+\lambda\Delta t)\hat{M}\end{pmatrix}\begin{pmatrix}\boldsymbol{U}^n_{\epsilon,h}\\ \boldsymbol{W}^n_{\epsilon,h}\end{pmatrix}=\begin{pmatrix}\hat{M}\boldsymbol{U}^{n-1}_{\epsilon,h}+\lambda\Delta t\boldsymbol{f}_h\\ \boldsymbol{0}\end{pmatrix}.\tag{3.16}$$

Since a finite-dimensional linear system is being considered, existence follows from uniqueness. Uniqueness is proved by an induction argument. Indeed, assume that $\boldsymbol{U}_{\epsilon,h}^{n-1}$ is unique (note that $\boldsymbol{U}_{\epsilon,h}^{0}$ is unique) and suppose that $(\boldsymbol{U}_{\epsilon,h}^{n}, \boldsymbol{W}_{\epsilon,h}^{n})^{\mathrm{T}}$ and $(\tilde{\boldsymbol{U}}_{\epsilon,h}^{n}, \tilde{\boldsymbol{W}}_{\epsilon,h}^{n})^{\mathrm{T}}$ are two solutions of (3.16). Let $\boldsymbol{\theta}_{U,\epsilon,h}^{n} = \boldsymbol{U}_{\epsilon,h}^{n} - \tilde{\boldsymbol{U}}_{\epsilon,h}^{n}$ and $\boldsymbol{\theta}_{W,\epsilon,h}^{n} = \boldsymbol{W}_{\epsilon,h}^{n} - \tilde{\boldsymbol{W}}_{\epsilon,h}^{n}$. It follows that

$$\begin{aligned} 0 &= ([\boldsymbol{\theta}_{W,\epsilon,h}^{n}]^{\mathrm{T}}[\boldsymbol{\theta}_{U,\epsilon,h}^{n}]^{\mathrm{T}})\begin{pmatrix} (1+\lambda\Delta t)\hat{M} & \Delta t K \\ (1+\lambda\Delta t)A_{\epsilon,h}(U_{\epsilon,h}^{n-1}) & -(1+\lambda\Delta t)\hat{M} \end{pmatrix}\begin{pmatrix} \boldsymbol{\theta}_{U,\epsilon,h}^{n} \\ \boldsymbol{\theta}_{W,\epsilon,h}^{n} \end{pmatrix} \\ &= (1+\lambda\Delta t)[\boldsymbol{\theta}_{U,\epsilon,h}^{n}]^{\mathrm{T}} A_{\epsilon,h}(U_{\epsilon,h}^{n-1})\boldsymbol{\theta}_{U,\epsilon,h}^{n} + \Delta t[\boldsymbol{\theta}_{W,\epsilon,h}^{n}]^{\mathrm{T}} K\boldsymbol{\theta}_{W,\epsilon,h}^{n}. \end{aligned}$$

By the positive semidefiniteness of $A_{\epsilon,h}(U_{\epsilon,h}^{n-1})$ and K, it follows that $\boldsymbol{\theta}_{U,\epsilon,h}^{n}$ and $\boldsymbol{\theta}_{W,\epsilon,h}^{n}$ are constant vectors. Hence,

$$\begin{aligned} 0 &= ([\boldsymbol{\theta}_{U,\epsilon,h}^{n}]^{\mathrm{T}}[\boldsymbol{\theta}_{W,\epsilon,h}^{n}]^{\mathrm{T}})\begin{pmatrix} (1+\lambda\Delta t)\hat{M} & \Delta t K \\ -(1+\lambda\Delta t)A_{\epsilon,h}(U_{\epsilon,h}^{n-1}) & (1+\lambda\Delta t)\hat{M} \end{pmatrix}\begin{pmatrix} \boldsymbol{\theta}_{U,\epsilon,h}^{n} \\ \boldsymbol{\theta}_{W,\epsilon,h}^{n} \end{pmatrix} \\ &= (1+\lambda\Delta t)\{[\boldsymbol{\theta}_{U,\epsilon,h}^{n}]^{\mathrm{T}}\hat{M}\boldsymbol{\theta}_{U,\epsilon,h}^{n} + [\boldsymbol{\theta}_{W,\epsilon,h}^{n}]^{\mathrm{T}}\hat{M}\boldsymbol{\theta}_{W,\epsilon,h}^{n}\}. \end{aligned}$$

Since $\hat{M}$ is positive definite, it follows that $\boldsymbol{\theta}_{U,\epsilon,h}^{n} = \boldsymbol{\theta}_{W,\epsilon,h}^{n} = \mathbf{0}$. Hence, there exists a unique solution of $(\mathbf{P}_1^{\epsilon,h,\Delta t})$.

(e) This can be proved similarly to (c), with inequality (2.2) being used instead of (2.1).

(f) For $\epsilon > 0$, take $\{U_{\epsilon,h}^{n}\}_{n=1}^{N} \subset V^h$ to be the solution of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ for data

$$u_{0,\epsilon,h} = u_{0,h}.$$

The inequalities (2.7), (2.13), $|\boldsymbol{p}|_\epsilon \leqslant |\boldsymbol{p}| + \epsilon$ and $\frac{1}{2}(a+b)^2 \leqslant a^2 + b^2$ yield

$$\begin{aligned} \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) = \hat{J}_{\lambda,\epsilon,h}(u_{0,h}) &= (|\nabla u_{0,h}|_\epsilon, 1) + \frac{\lambda}{2}\|\nabla[\hat{\mathcal{G}}^h u_{0,h} - \mathcal{G}^h f]\|^2 \\ &\leqslant (|\nabla u_{0,h}|, 1) + \epsilon|\Omega| + \lambda\|u_{0,h}\|_{-h,h}^2 + \lambda\|f\|_{-h}^2 \\ &\leqslant |\Omega|^{\frac{1}{2}}\|u_{0,h}\|_{H^1(\Omega)} + \epsilon|\Omega| + \lambda C\|u_{0,h}\|_{H^1(\Omega)}^2 + \lambda\|f\|_{-1}^2. \end{aligned} \tag{3.17}$$

The inequalities (2.7), (3.10) and $-\frac{1}{4}a^2 - b^2 \leqslant -ab$ give that for $t \in [0, T]$,

$$\begin{aligned} \frac{\lambda}{4}\|U_{\epsilon,h}(t)\|_{-h,h}^2 &= \left(\frac{\lambda}{2}\|U_{\epsilon,h}(t)\|_{-h,h}^2 + \frac{\lambda}{2}\|f\|_{-h}^2\right) - \left(\frac{\lambda}{4}\|U_{\epsilon,h}(t)\|_{-h,h}^2 + \lambda\|f\|_{-h}^2\right) + \frac{\lambda}{2}\|f\|_{-h}^2 \\ &\leqslant \frac{\lambda}{2}\|\nabla[\hat{\mathcal{G}}^h U_{\epsilon,h}(t) - \mathcal{G}^h f]\|^2 + \frac{\lambda}{2}\|f\|_{-h}^2 \leqslant \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(t)) + \frac{\lambda}{2}\|f\|_{-h}^2 \\ &\leqslant \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) + \frac{\lambda}{2}\|f\|_{-1}^2. \end{aligned} \tag{3.18}$$

The inequalities (2.8) give that there exists $C \equiv C(u_{0,h}, f, \lambda, \epsilon, \Omega)$, which remains bounded as $\epsilon \downarrow 0$, such that

$$\|U_{\epsilon,h}\|_{C(0,T;H^1(\Omega))} \leqslant \frac{C}{h^2}.$$

Hence, there exist a subsequence of $\{U_{\epsilon,h}\}_{\epsilon>0}$, still denoted by $\{U_{\epsilon,h}\}_{\epsilon>0}$, and a piecewise linear in time function $U_h \in L^\infty(0,T;V^h)$ such that

$$U_{\epsilon,h}(s) \to U_h(s) \;\text{ in } V^h \quad \text{as } \epsilon \downarrow 0.$$

We define $\{U_h^n\}_{n=1}^N \subset V^h$ by

$$U_h^n := U_h(t_n) \quad \forall\, n = 0,\ldots,N.$$

In particular,

$$\forall\, n = 0,\ldots,N, \quad U_{\epsilon,h}^n \to U_h^n \;\text{ in } V^h \quad \text{as } \epsilon \downarrow 0. \tag{3.19}$$

Take $v_h \in V^h$. Lemma 3.6 identifies the $\epsilon \downarrow 0$ limit of each term in inequality (3.4) as being the corresponding term in the inequality (3.3) for $(\mathbf{P}^{h,\Delta t})$.

LEMMA 3.6 For $v_h \in V^h$ and $n = 1,\ldots,N$,

(i) $\lim_{\epsilon\downarrow 0}(d_t U_{\epsilon,h}^n, \hat{\mathcal{G}}^h v_h)^h = (d_t U_h^n, \hat{\mathcal{G}}^h v_h)^h$, (ii) $\lim_{\epsilon\downarrow 0}(d_t U_{\epsilon,h}^n, \hat{\mathcal{G}}^h U_{\epsilon,h}^n)^h = (d_t U_h^n, \hat{\mathcal{G}}^h U_h^n)^h$,

(iii) $\lim_{\epsilon\downarrow 0} \hat{J}_{\lambda,\epsilon,h}(v_h) = \hat{J}_{\lambda,h}(v_h)$, (iv) $\lim_{\epsilon\downarrow 0} \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}^n) = \hat{J}_{\lambda,h}(U_h^n)$.

Proof.

(i) The inequality (2.13) and the limits (3.19) give that

$$\forall\, j = 0,\ldots,N, \quad |(U_{\epsilon,h}^j - U_h^j, \hat{\mathcal{G}}^h v_h)^h| \leqslant \|U_{\epsilon,h}^j - U_h^j\|_{-h,h}\|v_h\|_{-h,h} \leqslant C\|U_{\epsilon,h}^j - U_h^j\|\|v_h\| \to 0 \quad \text{as } \epsilon \downarrow 0.$$

(ii) The inequality (2.13) and the limits (3.19) give that for $j, n = 0,\ldots,N$,

$$\begin{aligned} |(U_{\epsilon,h}^j, \hat{\mathcal{G}}^h U_{\epsilon,h}^n)^h - (U_h^j, \hat{\mathcal{G}}^h U_h^n)^h| &\leqslant |(U_{\epsilon,h}^j - U_h^j, \hat{\mathcal{G}}^h U_{\epsilon,h}^n)^h| + |(U_h^j, \hat{\mathcal{G}}^h [U_{\epsilon,h}^n - U_h^n])^h| \\ &\leqslant C[\|U_{\epsilon,h}^j - U_h^j\|\|U_{\epsilon,h}^n\| + \|U_h^j\|\|U_{\epsilon,h}^n - U_h^n\|] \\ &\to 0 \quad \text{as } \epsilon \downarrow 0. \end{aligned}$$

(iii) Follows from $|\,|\boldsymbol{p}|_\epsilon - |\boldsymbol{p}|\,| \leqslant \epsilon$:

$$\left|\int_0^s (|\nabla v_h(t)|_\epsilon - |\nabla v_h(t)|, 1)\mathrm{d}t\right| \leqslant \epsilon T|\Omega| \to 0 \quad \text{as } \epsilon \downarrow 0.$$

(iv) The inequality $|\,|\boldsymbol{p}|_\epsilon - |\boldsymbol{p}|\,| \leqslant \epsilon$ and limits (3.19) give that

$$\begin{aligned} \forall\, n = 1,\ldots,N, \quad & |(|\nabla U_{\epsilon,h}^n|_\epsilon - |\nabla U_h^n|, 1)| \\ &\leqslant (||\nabla U_{\epsilon,h}^n|_\epsilon - |\nabla U_{\epsilon,h}^n||, 1) + (||\nabla U_{\epsilon,h}^n| - |\nabla U_h^n||, 1) \\ &\leqslant \epsilon|\Omega| + |\Omega|^{\frac{1}{2}}\|U_{\epsilon,h}^n - U_h^n\|_{H^1(\Omega)} \to 0 \quad \text{as } \epsilon \downarrow 0. \end{aligned}$$

From the proof of (ii),

$$\forall\, n = 1, \ldots, N, \quad |(U^n_{\epsilon,h}, \hat{\mathcal{G}}^h U^n_{\epsilon,h})^h - (U^n_h, \hat{\mathcal{G}}^h U^n_h)^h| \to 0 \quad \text{as } \epsilon \downarrow 0.$$

Also, the inequalities (2.7), (2.13) and limits (3.19) give that

$$\forall\, n = 1, \ldots, N, \quad |(\nabla \hat{\mathcal{G}}^h [U^n_{\epsilon,h} - U^n_h], \nabla \mathcal{G}^h f)| \leqslant \|U^n_{\epsilon,h} - U^n_h\|_{-h,h} \|f\|_{-h}$$
$$\leqslant C \|U^n_{\epsilon,h} - U^n_h\| \|f\|_{-1} \to 0 \quad \text{as } \epsilon \downarrow 0.$$

□

(g) This can be proved analogously to (b).

(h) Let $\{U^n_h\}_{n=1}^N \subset V^h$ be the solution of $(\mathbf{P}^{h,\Delta t})$. Taking $v_h = U^{n-1}_h$ in inequality (3.3) gives

$$\Delta t \|d_t U^n_h\|^2_{-h,h} + \hat{J}_{\lambda,h}(U^n_h) \leqslant \hat{J}_{\lambda,h}(U^{n-1}_h) \quad \forall\, n = 1, \ldots, N.$$

Summing and using equation (2.18) yields

$$\|U'_h\|^2_{L^2(0,t_n;\mathcal{F}^h)} + \hat{J}_{\lambda,h}(U^n_h) \leqslant \hat{J}_{\lambda,h}(u_{0,h}) \quad \forall\, n = 0, \ldots, N.$$

Proceeding as in (c) gives inequality (3.12). □

4. Convergence results

In this section, we address convergence as $\epsilon, h, \Delta t \downarrow 0$. In Section 4.1, we introduce spatially discrete (but continuous in time) analogues $(\mathbf{P}^{\epsilon,h})$ and $(\mathbf{P}^{h})$ of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ and $(\mathbf{P}^{h,\Delta t})$, respectively. We consider convergence as $\epsilon, h, \Delta t \downarrow 0$ in Section 4.2. Theorem 4.3 gives the convergence of the sequences of solutions of $(\mathbf{P}^{\epsilon,h})$ and $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ to the solutions of $(\mathbf{P}^{h})$ and $(\mathbf{P}^{h,\Delta t})$, respectively, as $\epsilon \downarrow 0$. Theorem 4.4 gives the convergence of the sequences of solutions of $(\mathbf{P}_2^{\epsilon,h,\Delta t})$ and $(\mathbf{P}^{h,\Delta t})$ to the solutions of $(\mathbf{P}^{\epsilon,h})$ and $(\mathbf{P}^{h})$, respectively, as $\Delta t \downarrow 0$. Theorem 4.6 gives the convergence of the sequences of solutions of $(\mathbf{P}^{\epsilon,h})$ and $(\mathbf{P}^{h})$ to the weak solutions of $(\mathbf{P}^{\epsilon})$ and $(\mathbf{P})$, respectively, as $h \downarrow 0$. Suboptimal rates for convergence as $\epsilon, \Delta t \downarrow 0$ are given in Section 4.3.

4.1 *Spatially discrete problems*

We consider the following discrete in space approximation to the fourth-order PDE (1.2):

$(\mathbf{P}^{\epsilon,h})$ for a.e. $t \in (0, T)$, find $u_{\epsilon,h}(t) \in V^h$ such that $u_{\epsilon,h}(0) = u_{0,\epsilon,h} \in V^h$ and

$$(\hat{\mathcal{G}}^h u'_{\epsilon,h}(t), \eta_h)^h + \left(\frac{\nabla u_{\epsilon,h}(t)}{|\nabla u_{\epsilon,h}(t)|_\epsilon}, \nabla \eta_h\right) = -\lambda(\hat{\mathcal{G}}^h u_{\epsilon,h}(t) - \mathcal{G}^h f, \eta_h)^h \quad \forall\, \eta_h \in S^h. \tag{4.1}$$

We also consider the following discrete in space approximation to the unregularized analogue of the variational inequality (1.8):

($\mathbf{P}^{h}$) find $u_h \in H^1(0,T;V^h) \cap C(0,T;V^h)$ such that $u_h(0) = u_{0,h} \in V^h$ and for all $s \in [0,T]$,

$$\int_0^s (u_h'(t), \hat{\mathcal{G}}^h[v_h(t) - u_h(t)])^h \,\mathrm{d}t + \int_0^s [\hat{J}_{\lambda,h}(v_h(t)) - \hat{J}_{\lambda,h}(u_h(t))]\mathrm{d}t \geqslant 0$$
$$\forall\, v_h \in L^2(0,T;V^h). \tag{4.2}$$

The proofs of Lemmas 4.1 and 4.2 are omitted.

LEMMA 4.1 If $u_{\epsilon,h}\colon t \in (0,T) \to u_{\epsilon,h}(t) \in V^h$ is a solution of **($\mathbf{P}^{\epsilon,h}$)**, then there exists a constant $C \equiv C(u_{0,\epsilon,h}, f, \lambda, \epsilon, \Omega)$ such that

$$\|u_{\epsilon,h}\|_{C(0,T;V^h)}, \|U'_{\epsilon,h}\|_{L^2(0,T;V^h)} \leqslant \frac{C}{h^2},$$

and hence $u_{\epsilon,h} \in H^1(0,T;V^h) \cap C(0,T;V^h)$. Further, $u_{\epsilon,h}$ satisfies

$$\int_0^s (v_h'(t), \hat{\mathcal{G}}^h[v_h(t) - u_{\epsilon,h}(t)])^h \mathrm{d}t + \int_0^s [\hat{J}_{\lambda,\epsilon,h}(v_h(t)) - \hat{J}_{\lambda,\epsilon,h}(u_{\epsilon,h}(t))]\mathrm{d}t$$
$$\geqslant \frac{1}{2}[\|v_h(s) - u_{\epsilon,h}(s)\|^2_{-h,h} - \|v_h(0) - u_{0,\epsilon,h}\|^2_{-h,h}]$$
$$\forall\, v_h \in H^1(0,T;V^h) \cap C(0,T;V^h). \tag{4.3}$$

LEMMA 4.2

(i) The problem **($\mathbf{P}^{\epsilon,h}$)** has a unique solution $u_{\epsilon,h}$. Moreover, $u_{\epsilon,h}$ satisfies the stability estimate

$$\int_0^s \|u'_{\epsilon,h}(t)\|^2_{-h,h} \,\mathrm{d}t + \hat{J}_{\lambda,\epsilon,h}(u_{\epsilon,h}(s)) = \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) \quad \forall\, s \in [0,T]. \tag{4.4}$$

Let $u_{\epsilon,h,i}$ $(i = 1,2)$ be solutions of **($\mathbf{P}^{\epsilon,h}$)** for data $u_{0,\epsilon,h,i} \in V^h$, $f_i \in \mathcal{F}$. Then, for all $s \in [0,T]$,

$$\|u_{\epsilon,h,2}(s) - u_{\epsilon,h,1}(s)\|_{-h,h} \leqslant \|u_{0,\epsilon,h,2} - u_{0,\epsilon,h,1}\|_{-h,h} + \sqrt{\lambda T}\|f_2 - f_1\|_{-h}.$$

(ii) The problem **($\mathbf{P}^{h}$)** has a unique solution u_h. Moreover, u_h satisfies the stability estimate

$$\int_0^s \|u_h'(t)\|^2_{-h,h}\mathrm{d}t + \hat{J}_{\lambda,h}(u_h(s)) \leqslant \hat{J}_{\lambda,h}(u_{0,h}) \quad \forall\, s \in [0,T].$$

Let $u_{h,i}$ $(i = 1,2)$ be solutions of **($\mathbf{P}^{h}$)** for data $u_{0,h,i} \in V^h$, $f_i \in \mathcal{F}$. Then, for all $s \in [0,T]$,

$$\|u_{h,2}(s) - u_{h,1}(s)\|_{-h,h} \leqslant \|u_{0,h,2} - u_{0,h,1}\|_{-h,h} + \sqrt{\lambda T}\|f_2 - f_1\|_{-h,h}.$$

We note that it follows from the proof of Lemma 4.2 that $u_{\epsilon,h} \in H^1(0,T;V^h) \cap C(0,T;V^h)$ is the solution of **($\mathbf{P}^{\epsilon,h}$)** if, and only if, it satisfies (4.3).

Suppose that $u_{\epsilon,h} \in H^1(0,T;V^h) \cap C(0,T;V^h)$ is the solution of **($\mathbf{P}^{\epsilon,h}$)**. Taking $\eta_h = u'_{\epsilon,h}(t)$ in equation (4.1) and integrating over $t \in (0,s)$ give

$$\int_0^s \|u'_{\epsilon,h}(t)\|^2_{-h,h} \,\mathrm{d}t + \hat{J}_{\lambda,\epsilon,h}(u_{\epsilon,h}(s)) = \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) \quad \forall\, s \in [0,T]. \tag{4.5}$$

4.2 *Convergence as* $\epsilon, h, \Delta t \downarrow 0$

THEOREM 4.3 (Convergence as $\epsilon \downarrow 0$)

(i) Let $u_h \in H^1(0,T;V^h) \cap C(0,T;V^h)$ be the solution of $(\mathbf{P^h})$, and for $\epsilon > 0$ let $u_{\epsilon,h} \in H^1(0,T;V^h) \cap C(0,T;V^h)$ be the solution of $(\mathbf{P^{\epsilon,h}})$ for data $u_{0,\epsilon,h} = u_{0,h}$. Then

$$u_{\epsilon,h} \to u_h \quad \text{in } L^2(\Omega_T), \qquad u_{\epsilon,h} \rightharpoonup u_h \quad \text{in } L^2(0,s;V^h),$$

$$u'_{\epsilon,h} \rightharpoonup u'_h \quad \text{in } L^2(0,s;V^h), \quad u_{\epsilon,h}(s) \to u_h(s) \quad \text{in } V^h \qquad \text{as } \epsilon \downarrow 0.$$

(ii) Let $\{U_h^n\}_{n=1}^N \subset V^h$ be the solution of $(\mathbf{P^{h,\Delta t}})$, and for $\epsilon > 0$ let $\{U_{\epsilon,h}^n\}_{n=1}^N \subset V^h$ be the solution of $(\mathbf{P_2^{\epsilon,h,\Delta t}})$ for data $u_{0,\epsilon,h} = u_{0,h}$. Then

$$\forall\, n = 0, \dots, N, \quad U_{\epsilon,h}^n \to U_h^n \ \text{ in } V^h \quad \text{as } \epsilon \downarrow 0.$$

Proof. We omit the proof of (i), and (ii) follows from the proof of (f) in Section 3.4. □

THEOREM 4.4 (Convergence as $\Delta t \downarrow 0$)

(i) Let $u_{\epsilon,h}$ be the solution of $(\mathbf{P^{\epsilon,h}})$, and $U_{\epsilon,h}$ and $\hat{U}_{\epsilon,h}$ be the piecewise linear and piecewise constant in time interpolants of the solution $\{U_{\epsilon,h}^n\}_{n=0}^N$ of $(\mathbf{P_2^{\epsilon,h,\Delta t}})$. As $\Delta t \downarrow 0$,

$$U_{\epsilon,h}, \hat{U}_{\epsilon,h} \rightharpoonup u_{\epsilon,h} \ \text{ in } L^2(0,T;V^h), \quad U'_{\epsilon,h} \rightharpoonup u'_{\epsilon,h} \ \text{ in } L^2(0,T;V^h),$$

$$\text{for a.e. } s \in [0,T], \quad U_{\epsilon,h}(s), \hat{U}_{\epsilon,h}(s) \to u_{\epsilon,h}(s) \quad \text{in } V^h.$$

(ii) Let u_h be the solution of $(\mathbf{P^h})$ and U_h and $\hat{U}_h$ be the piecewise linear and piecewise constant in time interpolants of the solution $\{U_h^n\}_{n=0}^N$ of $(\mathbf{P^{h,\Delta t}})$. As $\Delta t \downarrow 0$,

$$U_h, \hat{U}_h \rightharpoonup u_h \ \text{ in } L^2(0,T;V^h), \quad U'_h \rightharpoonup u'_h \ \text{ in } L^2(0,T;V^h),$$

$$\text{for a.e. } s \in [0,T], \quad U_h(s), \hat{U}_h(s) \to u_h(s) \ \text{ in } V^h.$$

Proof. We prove (i) ((ii) can be proved analogously). From (3.18), for $\Delta t > 0$,

$$\frac{\lambda}{4}\|U_{\epsilon,h}(t)\|_{-h,h}^2 \leqslant \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) + \frac{\lambda}{2}\|f\|_{-h}^2 \quad \forall\, t \in [0,T].$$

Using the inequalities (2.7) and (2.8) yields

$$\|U_{\epsilon,h}\|_{C(0,T;V^h)} \leqslant \frac{C}{h^2}\sqrt{\frac{4}{\lambda}\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) + 2\|f\|_{-h}^2} \quad \forall\, \Delta t > 0.$$

Further, inequalities (2.7), (2.8) and (3.10) give that

$$\|U'_{\epsilon,h}\|_{L^2(0,T;V^h)} \leqslant \frac{C}{h^2}\sqrt{\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h})} \quad \forall\, \Delta t > 0.$$

For $1 \leqslant n \leqslant N$ and $t \in [t_{n-1}, t_n)$,

$$\|\hat{U}_{\epsilon,h}(t) - U_{\epsilon,h}(t)\|_{V^h} = \left\| \frac{t - t_n}{\Delta t}(U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}) \right\|_{V^h} \leqslant \|U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}\|_{V^h} \leqslant 2\|U_{\epsilon,h}\|_{C(0,T;V^h)}.$$

Thus,

$$\begin{aligned}\|\hat{U}_{\epsilon,h}\|_{L^\infty(0,T;V^h)} &\leqslant \|\hat{U}_{\epsilon,h} - U_{\epsilon,h}\|_{L^\infty(0,T;V^h)} + \|U_{\epsilon,h}\|_{L^\infty(0,T;V^h)} \\ &\leqslant \frac{3C}{h^2}\sqrt{\frac{4}{\lambda}\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}) + 2\|f\|^2_{-h}} \quad \forall\, \Delta t > 0.\end{aligned}$$

Hence, there exist subsequences of $\{U_{\epsilon,h}\}_{\Delta t>0}$ and $\{\hat{U}_{\epsilon,h}\}_{\Delta t>0}$, still denoted by $\{U_{\epsilon,h}\}_{\Delta t>0}$ and $\{\hat{U}_{\epsilon,h}\}_{\Delta t>0}$, and functions $\overline{u}_{\epsilon,h}, \hat{\overline{u}}_{\epsilon,h} \in L^\infty(0, T; V^h)$ such that $\overline{u}'_{\epsilon,h} \in L^2(0, T; V^h)$ and as $\Delta t \downarrow 0$,

$$\forall\, s \in [0, T], \quad \{U_{\epsilon,h}, \hat{U}_{\epsilon,h}\} \rightharpoonup \{\overline{u}_{\epsilon,h}, \hat{\overline{u}}_{\epsilon,h}\} \text{ in } L^2(0, s; V^h), \quad U'_{\epsilon,h} \rightharpoonup \overline{u}'_{\epsilon,h} \text{ in } L^2(0, s; V^h); \tag{4.6}$$

$$\text{for a.e. } s \in [0, T], \quad \{U_{\epsilon,h}(s), \hat{U}_{\epsilon,h}(s)\} \to \{\overline{u}_{\epsilon,h}(s), \hat{\overline{u}}_{\epsilon,h}(s)\} \text{ in } V^h. \tag{4.7}$$

Further,

$$\begin{aligned}\|U_{\epsilon,h} - \hat{U}_{\epsilon,h}\|^2_{L^2(0,T;V^h)} &= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \left\| \frac{t - t_n}{\Delta t}(U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}) \right\|^2_{V^h} \mathrm{d}t = \frac{\Delta t}{3}\sum_{n=1}^{N}\|U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}\|^2_{V^h} \\ &= \frac{(\Delta t)^2}{3}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|U'_{\epsilon,h}\|^2_{V^h}\,\mathrm{d}t = \frac{(\Delta t)^2}{3}\|U'_{\epsilon,h}\|^2_{L^2(0,T;V^h)} \\ &\to 0 \quad \text{as } \Delta t \downarrow 0,\end{aligned}$$

i.e.

$$U_{\epsilon,h} - \hat{U}_{\epsilon,h} \to 0 \text{ in } L^2(0, T; V^h) \quad \text{as } \Delta t \downarrow 0.$$

Further, limits (4.6) yield

$$U_{\epsilon,h} - \hat{U}_{\epsilon,h} \rightharpoonup \overline{u}_{\epsilon,h} - \hat{\overline{u}}_{\epsilon,h} \text{ in } L^2(0, T; V^h) \quad \text{as } \Delta t \downarrow 0.$$

The uniqueness of weak limits gives that $\overline{u}_{\epsilon,h} = \hat{\overline{u}}_{\epsilon,h}$. Limits (4.6) and (4.7) give that as $\Delta t \downarrow 0$,

$$\forall\, s \in [0, T], \quad U_{\epsilon,h}, \hat{U}_{\epsilon,h} \rightharpoonup \overline{u}_{\epsilon,h} \text{ in } L^2(0, s; V^h), \quad U'_{\epsilon,h} \rightharpoonup \overline{u}'_{\epsilon,h} \text{ in } L^2(0, s; V^h); \tag{4.8}$$

$$\text{for a.e. } s \in [0, T], \quad U_{\epsilon,h}(s), \hat{U}_{\epsilon,h}(s) \to \overline{u}_{\epsilon,h}(s) \text{ in } V^h. \tag{4.9}$$

Lemma 4.5 identifies the $\Delta t \downarrow 0$ limit of each term in inequality (3.6) as being the corresponding term in the inequality (4.3) with $u_{\epsilon,h}$ replaced by $\overline{u}_{\epsilon,h}$.

LEMMA 4.5 For $v_h \in H^1(0,T;V^h) \cap C(0,T;V^h)$ and $s \in [0,T]$,

$$\text{(a)} \quad \lim_{\Delta t \downarrow 0} \int_0^s (v_h'(t), \hat{\mathcal{G}}^h U_{\epsilon,h}(t))^h \mathrm{d}t = \int_0^s (v_h'(t), \hat{\mathcal{G}}^h \overline{u}_{\epsilon,h}(t))^h \, \mathrm{d}t,$$

$$\text{(b)} \quad \lim_{\Delta t \downarrow 0} \int_0^s \hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t))\mathrm{d}t = \int_0^s \hat{J}_{\lambda,\epsilon,h}(\overline{u}_{\epsilon,h}(t))\mathrm{d}t,$$

$$\text{(c)} \quad \lim_{\Delta t \downarrow 0} \|v_h(s) - U_{\epsilon,h}(s)\|_{-h,h}^2 = \|v_h(s) - \overline{u}_{\epsilon,h}(s)\|_{-h,h}^2.$$

Proof.

(a) This follows from limits (4.8) and Lemma 2.3 (ii).

(b) Inequality (2.13) and limit (4.9) give that for a.e. $t \in [0,T]$,

$$\|U_{\epsilon,h}(t) - \overline{u}_{\epsilon,h}(t)\|_{-h,h} \to 0 \quad \text{as } \Delta t \downarrow 0. \tag{4.10}$$

Hence, the lower triangle inequality gives that, for a.e. $t \in [0,T]$,

$$|\|\nabla[\hat{\mathcal{G}}^h U_{\epsilon,h}(t) - \mathcal{G}^h f]\| - \|\nabla[\hat{\mathcal{G}}^h \overline{u}_{\epsilon,h}(t) - \mathcal{G}^h f]\|| \leqslant \|U_{\epsilon,h}(t) - \overline{u}_{\epsilon,h}(t)\|_{-h,h} \to 0 \quad \text{as } \Delta t \downarrow 0.$$

Lemma 2.1 and limit (4.9) give that for a.e. $t \in [0,T]$,

$$\begin{aligned} |\hat{J}_{0,\epsilon,h}(U_{\epsilon,h}(t)) - \hat{J}_{0,\epsilon,h}(\overline{u}_{\epsilon,h}(t))| &= |(|\nabla U_{\epsilon,h}(t)|_\epsilon - |\nabla \overline{u}_{\epsilon,h}(t)|_\epsilon, 1)| \\ &\leqslant (|\nabla[U_{\epsilon,h}(t) - \overline{u}_{\epsilon,h}(t)]|, 1) \\ &\leqslant |\Omega|^{\frac{1}{2}} \|U_{\epsilon,h}(t) - \overline{u}_{\epsilon,h}(t)\|_{H^1(\Omega)} \to 0 \quad \text{as } \Delta t \downarrow 0. \end{aligned}$$

Hence, for a.e. $t \in [0,T]$,

$$\hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t)) \to \hat{J}_{\lambda,\epsilon,h}(\overline{u}_{\epsilon,h}(t)) \quad \text{as } \Delta t \downarrow 0.$$

Using the dominated convergence theorem gives the stated result.

(c) The limit (4.10) and the lower triangle inequality yield that, for a.e. $s \in [0,T]$,

$$|\|v_h(s) - U_{\epsilon,h}(s)\|_{-h,h} - \|v_h(s) - \overline{u}_{\epsilon,h}(s)\|_{-h,h}| \leqslant \|\overline{u}_{\epsilon,h}(s) - U_{\epsilon,h}(s)\|_{-h,h} \to 0 \quad \text{as } \Delta t \downarrow 0.$$

□

An analogous argument to that in Section 4.6 in Elliott & Smitheman (2007) gives that $\overline{u}_{\epsilon,h}(0) = u_{0,\epsilon,h}$. Hence, $\overline{u}_{\epsilon,h}$ is a solution of **($\mathbf{P}^{\epsilon,h}$)** for data $\overline{u}_{\epsilon,h}(0) = u_{0,\epsilon,h} = u_{\epsilon,h}(0)$. The uniqueness of the solution of **($\mathbf{P}^{\epsilon,h}$)** gives that $\overline{u}_{\epsilon,h} = u_{\epsilon,h}$ and that the whole sequences $\{U_{\epsilon,h}\}_{\Delta t>0}$ and $\{\hat{U}_{\epsilon,h}\}_{\Delta t>0}$ satisfy (4.8) and (4.9). □

THEOREM 4.6 (Convergence as $h \downarrow 0$ for $d = 2$) (i) Assume that f, $u_{0,\epsilon}$, f_h, $u_{0,\epsilon,h}$ satisfy the following:

(a) $f \in \mathcal{F}, u_{0,\epsilon} \in \mathrm{BV}(\Omega) \cap \mathcal{F}$;
(b) $f_h, u_{0,\epsilon,h} \in V^h$ for $h > 0$;
(c) $\{J_{0,\epsilon,h}(u_{0,\epsilon,h}, f_h)\}_{h>0}, \{\|f_h\|\}_{h>0}$ are bounded;
(d) $\{u_{0,\epsilon,h}, f_h\} \to \{u_{0,\epsilon}, f\}$ in $\mathcal{F}$ as $h \downarrow 0$.

Let u_ϵ be the weak solution of **($\mathbf{P}^\epsilon$)** for data $f, u_{0,\epsilon}$ and $u_{\epsilon,h}$ be the solution of **($\mathbf{P}^{\epsilon,h}$)** for data $f_h, u_{0,\epsilon,h}$. As $h \downarrow 0$,

$$u_{\epsilon,h} \rightharpoonup u_\epsilon \ \text{ in } L^2(\Omega_T), \quad u_{\epsilon,h} \overset{\star}{\rightharpoonup} u_\epsilon \ \text{ in } L^\infty(0,T;L^2(\Omega)), \quad u'_{\epsilon,h} \rightharpoonup u'_\epsilon \ \text{ in } L^2(0,T;\mathcal{F});$$
$$\text{for a.e. } s \in [0,T], \quad u_{\epsilon,h}(s) \to u_\epsilon(s) \in \mathrm{BV}(\Omega) \ \text{ in } L^p(\Omega) \text{ for all } p \in [1,2)$$
$$\text{and} \quad u_{\epsilon,h}(s) \rightharpoonup u_\epsilon(s) \ \text{ in } L^2(\Omega).$$

(ii) Assume that $f, u_0, f_h, u_{0,h}$ satisfy the following:

(a) $f \in \mathcal{F}, u_0 \in \mathrm{BV}(\Omega) \cap \mathcal{F}$; (c) $\{J_{0,h}(u_{0,h}, f_h)\}_{h>0}, \{\| f_h \|\}_{h>0}$ are bounded;

(b) $f_h, u_{0,h} \in V^h$ for $h > 0$; (d) $\{u_{0,h}, f_h\} \to \{u_0, f\}$ in $\mathcal{F}$ as $h \downarrow 0$.

Let u be the weak solution of **(P)** for data f, u_0 and u_h be the solution of **($\mathbf{P}^h$)** for data $f_h, u_{0,h}$. As $h \downarrow 0$,

$$u_h \rightharpoonup u \ \text{ in } L^2(\Omega_T), \quad u_h \overset{\star}{\rightharpoonup} u \ \text{ in } L^\infty(0,T;L^2(\Omega)), \quad u'_h \rightharpoonup u' \ \text{ in } L^2(0,T;\mathcal{F});$$
$$\text{for a.e. } s \in [0,T], \quad u_h(s) \to u(s) \in \mathrm{BV}(\Omega) \ \text{ in } L^p(\Omega) \text{ for all } p \in [1,2)$$
$$\text{and} \quad u_h(s) \rightharpoonup u(s) \ \text{ in } L^2(\Omega).$$

Proof. We prove (i) ((ii) can be proved analogously). As in the proof of Theorem 4.4 (i), for $h > 0$

$$\frac{\lambda}{4}\|u_{\epsilon,h}(t)\|^2_{-h,h} \leqslant \hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}, f_h) + \frac{\lambda}{2}\| f_h \|^2_{-h} \quad \forall t \in [0,T]$$
$$\Rightarrow \|u_{\epsilon,h}\|_{C(0,T;\mathcal{F})} \leqslant C\sqrt{\frac{4}{\lambda}\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}, f_h) + 2\| f_h \|^2_{-h}}.$$

Further, inequalities (2.7), (2.8) and (4.5) give that

$$\|u'_{\epsilon,h}\|_{L^2(0,T;\mathcal{F})} \leqslant C\sqrt{\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}, f_h)} \quad \forall h > 0.$$

By a similar argument to that in Section 4.5 of Elliott & Smitheman (2007), there exists $C \equiv C(\Omega)$ such that

$$\|u_{\epsilon,h}\|_{C(0,T;\mathrm{BV}(\Omega))} \leqslant (C+1)\hat{J}_{\lambda,\epsilon,h}(u_{0,\epsilon,h}, f_h).$$

It follows that there exist a subsequence of $\{u_{\epsilon,h}\}_{h>0}$, still denoted by $\{u_{\epsilon,h}\}_{h>0}$, and a function $\overline{u}_\epsilon \in L^\infty(0,T;\mathrm{BV}(\Omega)) \cap L^\infty(0,T;\mathcal{F})$ such that $\overline{u}'_\epsilon \in L^2(0,T;\mathcal{F})$ and as $h \downarrow 0$,

$$\forall s \in [0,T], \quad \{u_{\epsilon,h}, u'_{\epsilon,h}\} \rightharpoonup \{\overline{u}_\epsilon, \overline{u}'_\epsilon\} \text{ in } L^2(0,s;\mathcal{F}), \quad u_{\epsilon,h} \rightharpoonup \overline{u}_\epsilon \text{ in } L^2(\Omega_T); \tag{4.11}$$

$$\text{for a.e. } s \in [0,T], \quad u_{\epsilon,h}(s) \to \overline{u}_\epsilon(s) \in \mathrm{BV}(\Omega) \text{ in } L^p(\Omega) \text{ for all } p \in [1,2), \tag{4.12}$$

$$u_{\epsilon,h}(s) \rightharpoonup \overline{u}_\epsilon(s) \text{ in } L^2(\Omega) \quad \text{and} \quad u_{\epsilon,h}(s) \rightharpoonup \overline{u}_\epsilon(s) \text{ in } \mathcal{F}. \tag{4.13}$$

From Section 4.6 of Elliott & Smitheman (2007), it is sufficient to consider $v \in C^1(0,T;C^2(\Omega)) \cap C^1(0,T;\mathcal{V})$. For $h > 0$, define $v_h\colon t \in [0,T] \to v_h(t) \in V^h$ by

$$v_h(t) := P^h v(t) \quad \forall\, t \in [0,T].$$

Inequalities (2.15) yield

$$\|v_h\|_{C(0,T;H^1(\Omega))} = \sup_{t\in[0,T]} \|P^h v(t)\|_{H^1(\Omega)} \leqslant C \sup_{t\in[0,T]} \|v(t)\|_{H^1(\Omega)} = C\|v\|_{C(0,T;H^1(\Omega))},$$

$$\|v_h'\|^2_{L^2(0,T;H^1(\Omega))} = \int_0^T \|P^h v'(t)\|^2_{H^1(\Omega)}\,\mathrm{d}t \leqslant C\int_0^T \|v'(t)\|^2_{H^1(\Omega)}\,\mathrm{d}t = C\|v'\|^2_{L^2(0,T;H^1(\Omega))}.$$

Hence, $v_h \in H^1(0,T;V^h) \cap C(0,T;V^h)$ for $h > 0$. Inequality (2.14) gives

$$\begin{aligned}\|v - v_h\|_{C(0,T;H^1(\Omega))} &= \sup_{t\in[0,T]} \|v(t) - v_h(t)\|_{H^1(\Omega)} = \sup_{t\in[0,T]} \|(I - P^h)v(t)\|_{H^1(\Omega)}\\ &\leqslant Ch \sup_{t\in[0,T]} \|v(t)\|_{H^2(\Omega)} = Ch\|v\|_{C(0,T;H^2(\Omega))}.\end{aligned}$$

Similarly,

$$\|v' - v_h'\|_{C(0,T;H^1(\Omega))} \leqslant Ch\|v'\|_{C(0,T;H^2(\Omega))}.$$

Inequality (2.4) gives

$$\|v - v_h\|_{C^1(0,T;\mathcal{F})},\ \|v - v_h\|_{C^1(0,T;H^1(\Omega))},\ \|v' - v_h'\|_{C^1(0,T;\mathcal{F})},\ \|v' - v_h'\|_{C^1(0,T;H^1(\Omega))} \to 0 \quad \text{as } h \downarrow 0. \tag{4.14}$$

Lemma 2.4 (i), (iii), assumptions (c), (d) and limits (4.14) yield

$$\forall\, t \in [0,T], \quad \lim_{h\downarrow 0} \|v_h(t) - f_h\|_{-h} = \lim_{h\downarrow 0} \|v_h(t) - f_h\|_{-1} = \|v(t) - f\|^2_{-1}, \tag{4.15}$$

$$\lim_{h\downarrow 0} \|\nabla[\hat{\mathcal{G}}^h v_h - \mathcal{G}v]\|_{L^2(\Omega_T)} = 0, \quad \lim_{h\downarrow 0} \|\nabla[\hat{\mathcal{G}}^h v_h' - \mathcal{G}v']\|_{L^2(\Omega_T)} = 0. \tag{4.16}$$

Lemma 4.7 identifies the $h \downarrow 0$ limit of each term in inequality (4.3) as being (up to inequality) the corresponding term in inequality (1.9) with u_ϵ replaced by $\overline{u}_\epsilon$.

LEMMA 4.7 For $v \in C^1(0,T;C^2(\Omega)) \cap C^1(0,T;\mathcal{V})$, $v_h(t) := P^h v(t)$ and $s \in [0,T]$,

$$\text{(a)} \quad \lim_{h\downarrow 0} \int_0^s (v_h'(t), \hat{\mathcal{G}}^h v_h(t))^h \mathrm{d}t = \int_0^s (v'(t), \mathcal{G}v(t))\mathrm{d}t,$$

$$\text{(b)} \quad \lim_{h\downarrow 0} \int_0^s (v_h'(t), \hat{\mathcal{G}}^h u_{\epsilon,h}(t))^h \mathrm{d}t = \int_0^s (v'(t), \mathcal{G}\overline{u}_\epsilon(t))\mathrm{d}t,$$

$$\text{(c)} \quad \lim_{h\downarrow 0} \int_0^s \hat{J}_{\lambda,\epsilon,h}(v_h(t),\, f_h)\mathrm{d}t = \int_0^s J_{\lambda,\epsilon}(v(t),\, f)\mathrm{d}t,$$

(d) $\liminf_{h\downarrow 0} \int_0^s \hat{J}_{\lambda,\epsilon,h}(u_{\epsilon,h}(t),\ f_h)\mathrm{d}t \geqslant \int_0^s J_{\lambda,\epsilon}(\overline{u}_\epsilon(t),\ f)\mathrm{d}t,$

(e) $\liminf_{h\downarrow 0} \|v_h(s) - u_{\epsilon,h}(s)\|^2_{-h,h} \geqslant \|v(s) - \overline{u}_\epsilon(s)\|^2_{-1},$

(f) $\lim_{h\downarrow 0} \|v_h(0) - u_{0,\epsilon,h}\|^2_{-h,h} = \|v(0) - u_{0,\epsilon}\|^2_{-1}.$

Proof.

(a) Inequalities (2.7) and limits (4.14), (4.16) yield

$$\left|\int_0^s [(v_h'(t), \hat{\mathcal{G}}^h v_h(t))^h - (v'(t), \mathcal{G}v(t))]\mathrm{d}t\right|$$
$$\leqslant \left|\int_0^s (\nabla[\hat{\mathcal{G}}^h v_h'(t) - \mathcal{G}v'(t)], \nabla\hat{\mathcal{G}}^h v_h(t))\mathrm{d}t\right| + \left|\int_0^s (\nabla\mathcal{G}v'(t), \nabla[\hat{\mathcal{G}}^h v_h(t) - \mathcal{G}v(t)])\mathrm{d}t\right|$$
$$\leqslant \|\nabla[\hat{\mathcal{G}}^h v_h' - \mathcal{G}v']\|_{L^2(\Omega_T)}\|v_h\|_{L^2(0,T;\mathcal{F}^h)} + \|\nabla[\hat{\mathcal{G}}^h v_h - \mathcal{G}v]\|_{L^2(\Omega_T)}\|v'\|_{L^2(0,T;\mathcal{F})}$$
$$\to 0 \quad \text{as } h \downarrow 0.$$

(b) Define T_1, T_2 and T_3 by

$$\left|\int_0^s [(v_h'(t), \hat{\mathcal{G}}^h u_{\epsilon,h}(t))^h - (v'(t), \mathcal{G}\overline{u}_\epsilon(t))]\mathrm{d}t\right|$$
$$\leqslant \left|\int_0^s (\nabla[\hat{\mathcal{G}}^h v_h'(t) - \mathcal{G}v'(t)], \nabla\hat{\mathcal{G}}^h u_{\epsilon,h}(t))\mathrm{d}t\right| + \left|\int_0^s (v'(t), [\hat{\mathcal{G}}^h - \mathcal{G}]u_{\epsilon,h}(t))\mathrm{d}t\right|$$
$$+ \left|\int_0^s (v'(t), \mathcal{G}[u_{\epsilon,h}(t) - \overline{u}_\epsilon(t)])\mathrm{d}t\right|$$
$$=: T_1 + T_2 + T_3.$$

Inequalities (2.7), (2.8) and limits (4.11), (4.16) give

$$T_1 \leqslant \|\nabla[\hat{\mathcal{G}}^h v_h' - \mathcal{G}v']\|_{L^2(\Omega_T)}\|u_{\epsilon,h}\|_{L^2(0,T;\mathcal{F}^h)} \to 0 \quad \text{as } h \downarrow 0.$$

Inequality (2.12) and limits (4.11) yield

$$T_2 \leqslant \|(\hat{\mathcal{G}}^h - \mathcal{G})u_{\epsilon,h}\|_{L^2(\Omega_T)}\|v'\|_{L^2(\Omega_T)} \leqslant Ch\|u_{\epsilon,h}\|_{L^2(\Omega_T)}\|v'\|_{L^2(\Omega_T)} \to 0 \quad \text{as } h \downarrow 0.$$

Lemma 2.2 (ii) and limits (4.11) give that $T_3 \to 0$ as $h \downarrow 0$.

(c) Lemma 2.4 (iv), limits (4.14) and assumptions (c), (d) give that, for all $s \in [0, T]$,

$$\lim_{h\downarrow 0} \int_0^s \|\nabla[\hat{\mathcal{G}}^h v_h(t) - \mathcal{G}^h f_h]\|^2 \mathrm{d}t = \lim_{h\downarrow 0} \int_0^s \|v_h(t) - f_h\|^2_{-1}\,\mathrm{d}t = \int_0^s \|v(t) - f\|^2_{-1}\,\mathrm{d}t.$$

Lemma 2.1 and the second limit in (4.14) give that

$$\forall\, t \in [0,T], \quad |\hat{J}_{0,\epsilon,h}(v_h(t),\, f_h) - J_{0,\epsilon}(v(t),\, f)| = |(|\nabla v_h(t)|_\epsilon - |\nabla v(t)|_\epsilon, 1)| \leqslant (|\nabla[v_h(t) - v(t)]|, 1) \leqslant |\Omega|^{\frac{1}{2}} \|v_h(t) - v(t)\|_{H^1(\Omega)} \to 0 \quad \text{as } h \downarrow 0.$$

Using limit (4.15) and the dominated convergence theorem gives the stated result.

(d) Lemma 2.4 (iv), limits (4.11), (4.13), assumptions (c), (d), the lower semicontinuity of a norm with respect to weak convergence and Fatou's lemma give that, for all $s \in [0,T]$,

$$\liminf_{h\downarrow 0} \int_0^s \|\nabla[\hat{\mathcal{G}}^h u_{\epsilon,h}(t) - \mathcal{G}^h f_h]\|^2 \mathrm{d}t = \liminf_{h\downarrow 0} \int_0^s \|u_{\epsilon,h}(t) - f_h\|_{-1}^2 \,\mathrm{d}t \geqslant \int_0^s \liminf_{h\downarrow 0} \|u_{\epsilon,h}(t) - f_h\|_{-1}^2 \,\mathrm{d}t \geqslant \int_0^s \|\overline{u}_\epsilon(t) - f\|_{-1}^2 \mathrm{d}t.$$

The convexity of $J_{0,\epsilon,h}(\cdot,\, f_h) = J_{0,\epsilon}(\cdot,\, f)$ (Acar & Vogel, 1994, Theorem 2.4), limit (4.12) and Fatou's lemma give that, for all $s \in [0,T]$,

$$\liminf_{h\downarrow 0} \int_0^s J_{0,\epsilon,h}(u_{\epsilon,h},\, f_h(t))\mathrm{d}t = \liminf_{h\downarrow 0} \int_0^s J_{0,\epsilon}(u_{\epsilon,h}(t),\, f)\mathrm{d}t \geqslant \int_0^s \liminf_{h\downarrow 0} J_{0,\epsilon}(u_{\epsilon,h}(t),\, f)\mathrm{d}t \geqslant \int_0^s J_{0,\epsilon}(\overline{u}_\epsilon(t),\, f)\mathrm{d}t.$$

(e) Lemma 2.4 (i), limits (4.13), (4.14) and the lower semicontinuity of a norm with respect to weak convergence give that, for a.e. $s \in [0,T]$,

$$\liminf_{h\downarrow 0} \|u_{\epsilon,h}(s) - v_h(s)\|_{-h,h}^2 = \liminf_{h\downarrow 0} \|u_{\epsilon,h}(s) - v_h(s)\|_{-1}^2 \geqslant \|\overline{u}_\epsilon(s) - v(s)\|_{-1}^2.$$

(f) By a similar argument to that in Section 4.5 of Elliott & Smitheman (2007), there exists $C \equiv C(\Omega)$ such that

$$\|u_{0,\epsilon,h}\|_{\mathrm{BV}(\Omega)} \leqslant (C+1)\hat{J}_{0,\epsilon,h}(u_{0,\epsilon,h},\, f_h) \quad \forall\, h > 0.$$

The continuous imbedding $\mathrm{BV}(\Omega) \hookrightarrow L^2(\Omega)$ gives that $\{\|u_{0,\epsilon,h}\|\}_{h>0}$ is bounded. Hence, the second limit in (4.14) gives that $\{\|v_h(0) - u_{0,\epsilon,h}\|\}_{h>0}$ is bounded. Lemma 2.4 (i), limits (4.14) and assumption (d) yield

$$\lim_{h\downarrow 0} \|v_h(0) - u_{0,\epsilon,h}\|_{-h,h}^2 = \lim_{h\downarrow 0} \|v_h(0) - u_{0,\epsilon,h}\|_{-1}^2 = \|v(0) - u_{0,\epsilon}\|_{-1}^2. \qquad \square$$

An analogous argument to that in Section 4.6 in Elliott & Smitheman (2007) gives that $\overline{u}_\epsilon \in C(0,T;\mathcal{F})$ and $\overline{u}_\epsilon(0) = u_{0,\epsilon}$. Hence, $\overline{u}_\epsilon$ is a weak solution of **($\mathbf{P}^\epsilon$)** for data $\overline{u}_\epsilon(0) = u_{0,\epsilon} = u_\epsilon(0)$. The uniqueness of the weak solution of **($\mathbf{P}^\epsilon$)** gives that $\overline{u}_\epsilon = u_\epsilon$ and that the whole sequence $\{u_{\epsilon,h}\}_{h>0}$ satisfies limits (4.12–4.13). $\square$

4.3 *Rates of convergence as* $\epsilon, \Delta t \downarrow 0$

THEOREM 4.8 (Rate of convergence as $\epsilon \downarrow 0$)

(i) Suppose that $2 \leqslant d \leqslant 3$, $u_{0,h} \in V^h$, $\{u_{0,\epsilon,h}\}_{\epsilon>0} \subset V^h$ and $f \in \mathcal{F}$. Let u_h, $\{u_{\epsilon,h}\}_{\epsilon>0}$ be the solutions of $(\mathbf{P}^{\boldsymbol{h}})$, $(\mathbf{P}^{\boldsymbol{\epsilon,h}})$ for data $u_{0,h}$ and f, $\{u_{0,\epsilon,h}\}_{\epsilon>0}$ and f. Then

$$\|u_h - u_{\epsilon,h}\|_{C(0,T;\mathcal{F}^h)} \leqslant \|u_{0,h} - u_{0,\epsilon,h}\|_{-h,h} + 2\sqrt{\epsilon T|\Omega|}.$$

Hence, if $u_{0,\epsilon,h} = u_{0,h}$ for $\epsilon > 0$, inequalities (2.7) and (2.8) give that

$$u_{\epsilon,h} \to u_h \quad \text{in } C(0,T;V^h) \quad \text{as } \epsilon \downarrow 0.$$

(ii) Suppose that $2 \leqslant d \leqslant 3$, $u_{0,h} \in V^h$, $\{u_{0,\epsilon,h}\}_{\epsilon>0} \subset V^h$ and $f \in \mathcal{F}$. Let U_h, $\{U_{\epsilon,h}\}_{\epsilon>0}$ be the piecewise linear in time interpolants of the solutions $\{U_h^n\}_{n=0}^N$, $\{\{U_{\epsilon,h}^n\}_{n=0}^N\}_{\epsilon>0}$ of $(\mathbf{P}^{\boldsymbol{h,\Delta t}})$, $(\mathbf{P}_2^{\boldsymbol{\epsilon,h,\Delta t}})$ for data $u_{0,h}$ and f, $\{u_{0,\epsilon,h}\}_{\epsilon>0}$ and f. Then

$$\|U_h - U_{\epsilon,h}\|_{C(0,T;\mathcal{F}^h)} \leqslant \|u_{0,h} - u_{0,\epsilon,h}\|_{-h,h} + 2\sqrt{\epsilon T|\Omega|}. \tag{4.17}$$

Hence, if $u_{0,\epsilon,h} = u_{0,h}$ for $\epsilon > 0$, inequalities (2.7) and (2.8) give that

$$U_{\epsilon,h} \to U_h \quad \text{in } C(0,T;V^h) \quad \text{as } \epsilon \downarrow 0.$$

Proof.

(i) The inequality $||\boldsymbol{p}|_\epsilon - |\boldsymbol{p}|| \leqslant \epsilon$ gives that $|\hat{J}_{\lambda,\epsilon,h}(\cdot) - \hat{J}_{\lambda,h}(\cdot)| \leqslant \epsilon|\Omega|$. Taking $v_h = u_h$, $v_h = u_{\epsilon,h}$ in inequalities (4.3), (4.2), respectively, and adding the resulting inequalities gives the stated result.

(ii) Taking $v_h = U_h^n$, $v_h = U_{\epsilon,h}^n$ in inequalities (3.4), (3.3), respectively, adding the resulting inequalities and using the inequality $|J_{\lambda,\epsilon,h}(\cdot) - J_{\lambda,h}(\cdot)| \leqslant \epsilon|\Omega|$ and equation (2.17) yields

$$\|U_{\epsilon,h}^n - U_h^n\|_{-h,h}^2 \leqslant \|U_{\epsilon,h}^{n-1} - U_h^{n-1}\|_{-h,h}^2 + 4\epsilon\Delta t|\Omega| \quad \forall\, 1 \leqslant n \leqslant N.$$

It follows that

$$\|U_{\epsilon,h}^n - U_h^n\|_{-h,h}^2 \leqslant \|u_{0,\epsilon} - u_0\|_{-h,h}^2 + 4\epsilon T|\Omega| \quad \forall\, 0 \leqslant n \leqslant N. \tag{4.18}$$

Taking square root yields

$$\|U_{\epsilon,h}^n - U_h^n\|_{-h,h} \leqslant \|u_{0,\epsilon} - u_0\|_{-h,h} + 2\sqrt{\epsilon T|\Omega|} \quad \forall\, 0 \leqslant n \leqslant N,$$

giving the stated result for $t = t_n$ $(0 \leqslant n \leqslant N)$.

To prove (4.17), suppose that $t \in [t_{n-1}, t_n]$ for some $1 \leqslant n \leqslant N$. There exists $\alpha \in [0,1]$ such that $t = \alpha t_{n-1} + (1-\alpha)t_n$. The convexity of $\|\cdot\|_{-1}^2$ (see Elliott & Smitheman, 2007,Section 3) and inequality (4.18) give

$$\begin{aligned}\|U_{\epsilon,h}(t) - U_h(t)\|_{-1}^2 &= \|(1-\alpha)(U_{\epsilon,h}^n - U_h^n) + \alpha(U_{\epsilon,h}^{n-1} - U_h^{n-1})\|_{-1}^2 \\ &\leqslant (1-\alpha)\|U_{\epsilon,h}^n - U_h^n\|_{-1}^2 + \alpha\|U_{\epsilon,h}^{n-1} - U_h^{n-1}\|_{-1}^2 \\ &\leqslant \|u_{0,\epsilon} - u_0\|_{-1}^2 + 4\epsilon T|\Omega|.\end{aligned}$$

Taking square root yields the stated result.

□

THEOREM 4.9 (Rate of convergence as $\Delta t \downarrow 0$)

(i) Assume that $u_{\epsilon,h}(0) \in V^h$, $\{U_{\epsilon,h}(0)\}_{\Delta t>0} \subset V^h$ and $f \in \mathcal{F}$. Let $u_{\epsilon,h}$ be the solution of **($\mathbf{P}^{\epsilon,h}$)** for data $u_{\epsilon,h}(0),\ f$, and $U_{\epsilon,h}$ be the piecewise linear in time interpolant of the solution $\{U^n_{\epsilon,h}\}_{n=0}^N$ of **($\mathbf{P}_2^{\epsilon,h,\Delta t}$)** for data $U_{\epsilon,h}(0),\ f$. There exists $C \equiv C(\{U_{\epsilon,h}(0)\}_{\Delta t>0}, u_{\epsilon,h}(0), \epsilon, \lambda, \Omega) > 0$ such that

$$\|u_{\epsilon,h} - U_{\epsilon,h}\|_{C(0,T;\mathcal{F}^h)} \leqslant \|u_{\epsilon,h}(0) - U_{\epsilon,h}(0)\|_{-h,h} + C(\Delta t)^{\frac{1}{2}}.$$

Hence, if $U_{\epsilon,h}(0) = u_{\epsilon,h}(0)$ for $\Delta t > 0$, inequalities (2.7) and (2.8) yield

$$U_{\epsilon,h} \to u_{\epsilon,h} \ \text{ in } C(0,T;V^h) \quad \text{as } \Delta t \downarrow 0.$$

(ii) Assume that $u_h(0) \in V^h$, $\{U_h(0)\}_{\Delta t>0} \subset V^h$ and $f \in \mathcal{F}$. Let u_h be the solution of **($\mathbf{P}^h$)** for data $u_h(0),\ f$, and U_h be the piecewise linear in time interpolant of the solution $\{U^n_h\}_{n=0}^N$ of **($\mathbf{P}^{h,\Delta t}$)** for data $U_h(0),\ f$. There exists $C \equiv C(\{U_h(0)\}_{\Delta t>0}, u_h(0), \lambda, \Omega) > 0$ such that

$$\|u_h - U_h\|_{C(0,T;\mathcal{F}^h)} \leqslant \|u_h(0) - U_h(0)\|_{-h,h} + C(\Delta t)^{\frac{1}{2}}.$$

Hence, if $U_h(0) = u_h(0)$ for $\Delta t > 0$, inequalities (2.7) and (2.8) yield

$$U_h \to u_h \ \text{ in } C(0,T;V^h) \quad \text{as } \Delta t \downarrow 0.$$

Proof. We prove (i) ((ii) can be proved analogously). Inequality (3.4) implies that for all $s \in [0,T]$ and all $v_h \in H^1(0,T;V^h) \cap C(0,T;V^h)$,

$$\int_0^s (U'_{\epsilon,h}(t), \hat{\mathcal{G}}^h[v_h(t) - \hat{U}_{\epsilon,h}(t)])^h \,\mathrm{d}t + \int_0^s [\hat{J}_{\lambda,\epsilon,h}(v_h(t)) - \hat{J}_{\lambda,\epsilon,h}(\hat{U}_{\epsilon,h}(t))]\mathrm{d}t \geqslant 0.$$

Letting $v_h = u_{\epsilon,h}$ in the last inequality and $v = \hat{U}_{\epsilon,h}$ in inequality (4.3) and adding the two resulting inequalities gives that for all $s \in [0,T]$,

$$\begin{aligned}
&\frac{1}{2}\|u_{\epsilon,h}(s) - U_{\epsilon,h}(s)\|^2_{-h,h} \\
&\quad \leqslant \frac{1}{2}\|u_{\epsilon,h}(0) - U_{\epsilon,h}(0)\|^2_{-h,h} + \int_0^s (u'_{\epsilon,h}(t) - U'_{\epsilon,h}(t), \hat{\mathcal{G}}^h[\hat{U}_{\epsilon,h}(t) - U_{\epsilon,h}(t)])^h \,\mathrm{d}t \\
&\quad \leqslant \frac{1}{2}\|u_{\epsilon,h}(0) - U_{\epsilon,h}(0)\|^2_{-h,h} + \|u'_{\epsilon,h} - U'_{\epsilon,h}\|_{L^2(0,s;\mathcal{F}^h)} \|\hat{U}_{\epsilon,h} - U_{\epsilon,h}\|_{L^2(0,s;\mathcal{F}^h)}.
\end{aligned}$$

Using inequality (3.10) gives that for all $s \in [0,T]$,

$$\begin{aligned}
\|\hat{U}_{\epsilon,h} - U_{\epsilon,h}\|^2_{L^2(0,s;\mathcal{F}^h)} &\leqslant \|\hat{U}_{\epsilon,h} - U_{\epsilon,h}\|^2_{L^2(0,T;\mathcal{F}^h)} = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \frac{t - t_n}{\Delta t}(U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}) \right\|^2_{-h,h} \mathrm{d}t \\
&= \frac{\Delta t}{3} \sum_{n=1}^N \|U^n_{\epsilon,h} - U^{n-1}_{\epsilon,h}\|^2_{-h,h} = \frac{(\Delta t)^2}{3} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|U'_{\epsilon,h}\|^2_{-h,h} \,\mathrm{d}t \\
&= \frac{(\Delta t)^2}{3} \|U'_{\epsilon,h}\|^2_{L^2(0,T;\mathcal{F}^h)} \leqslant \frac{(\Delta t)^2}{3} \hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(0)).
\end{aligned}$$

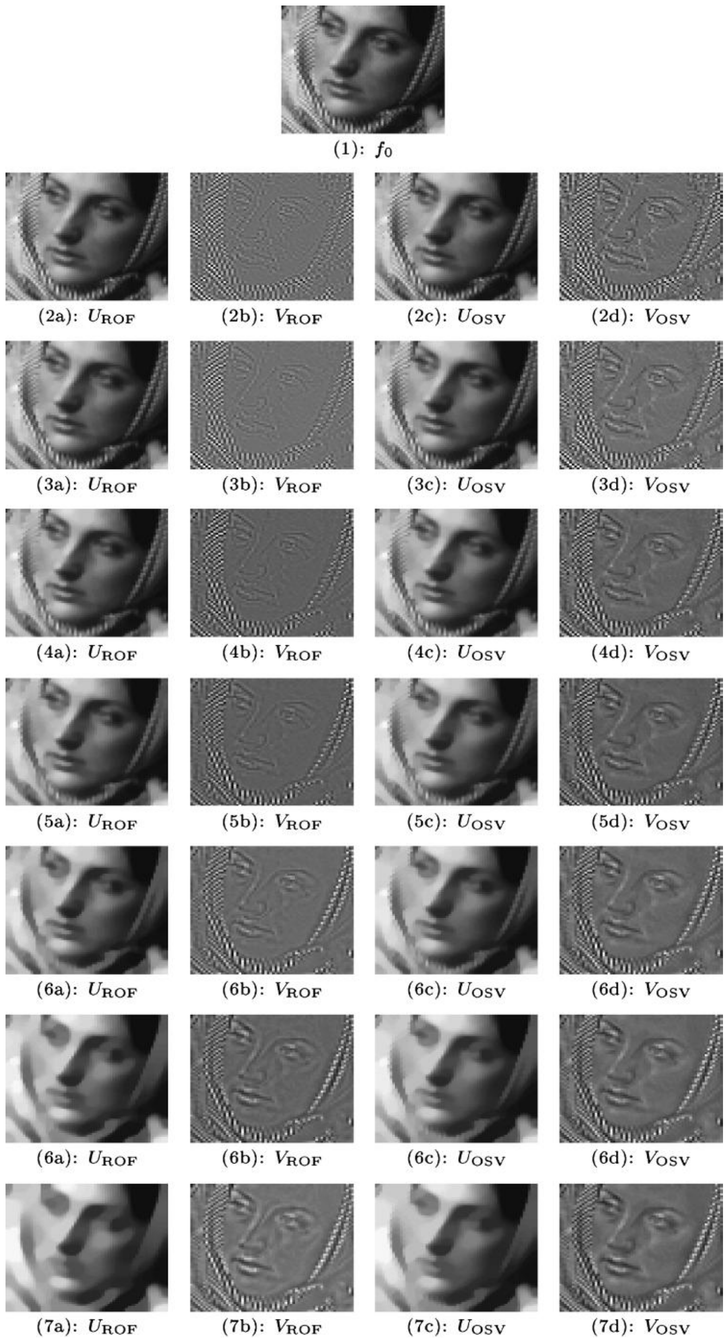

FIG. 1. The decomposition of the Barbara image.

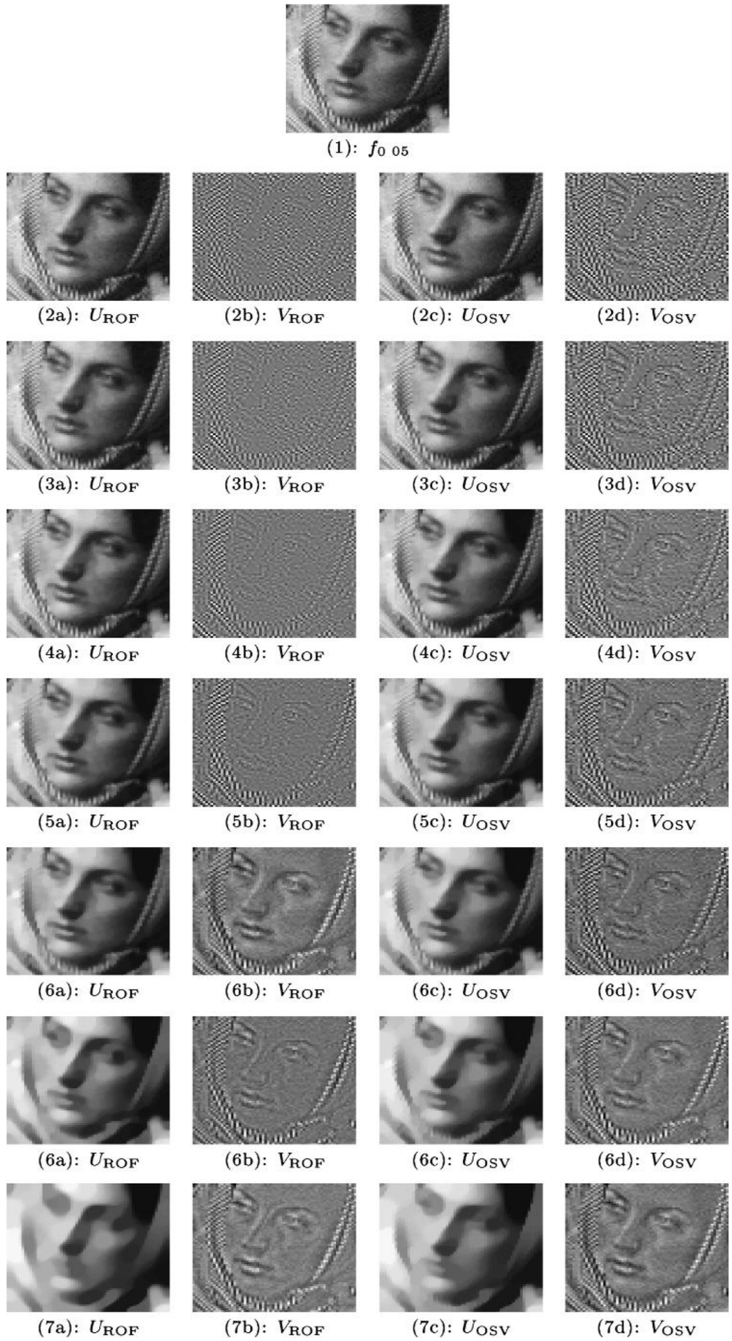

FIG. 2. The decomposition of the Barbara image degraded by additive Gaussian noise of mean 0 and standard deviation $\sigma = 0.05$.

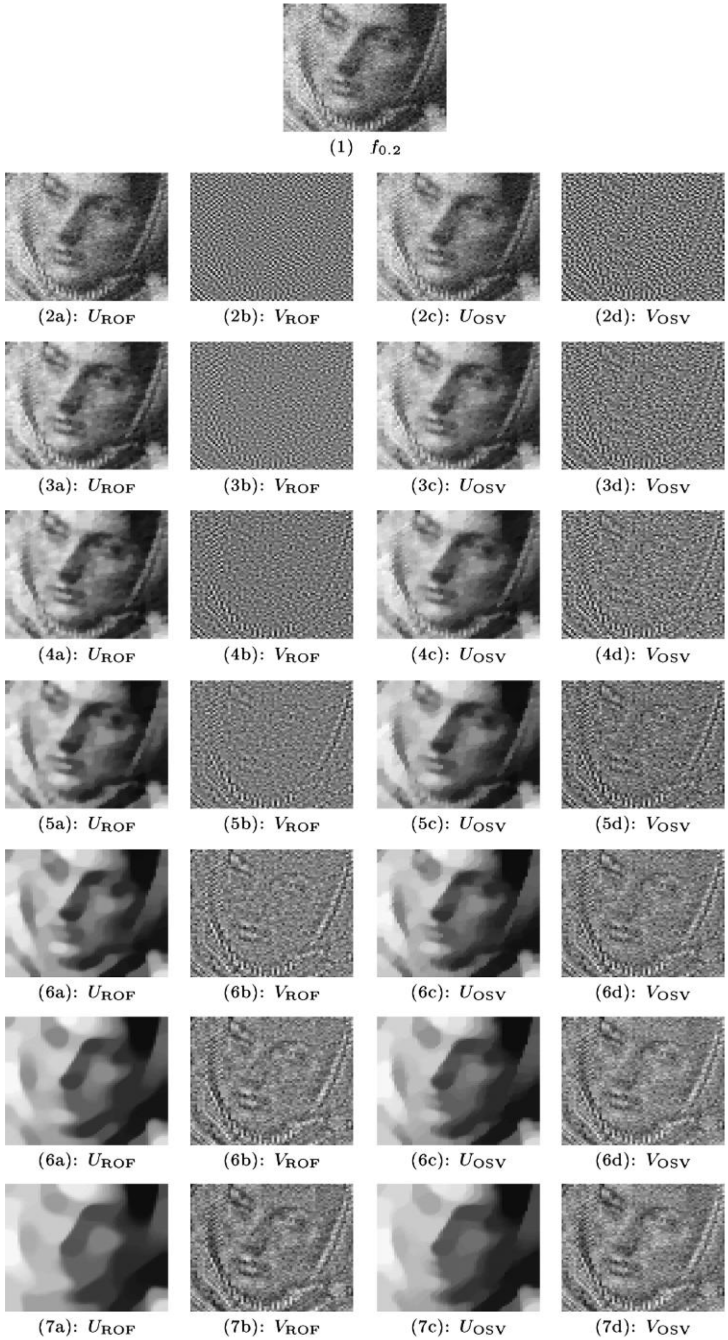

FIG. 3. The decomposition of the Barbara image degraded by additive Gaussian noise of mean 0 and standard deviation $\sigma = 0.2$.

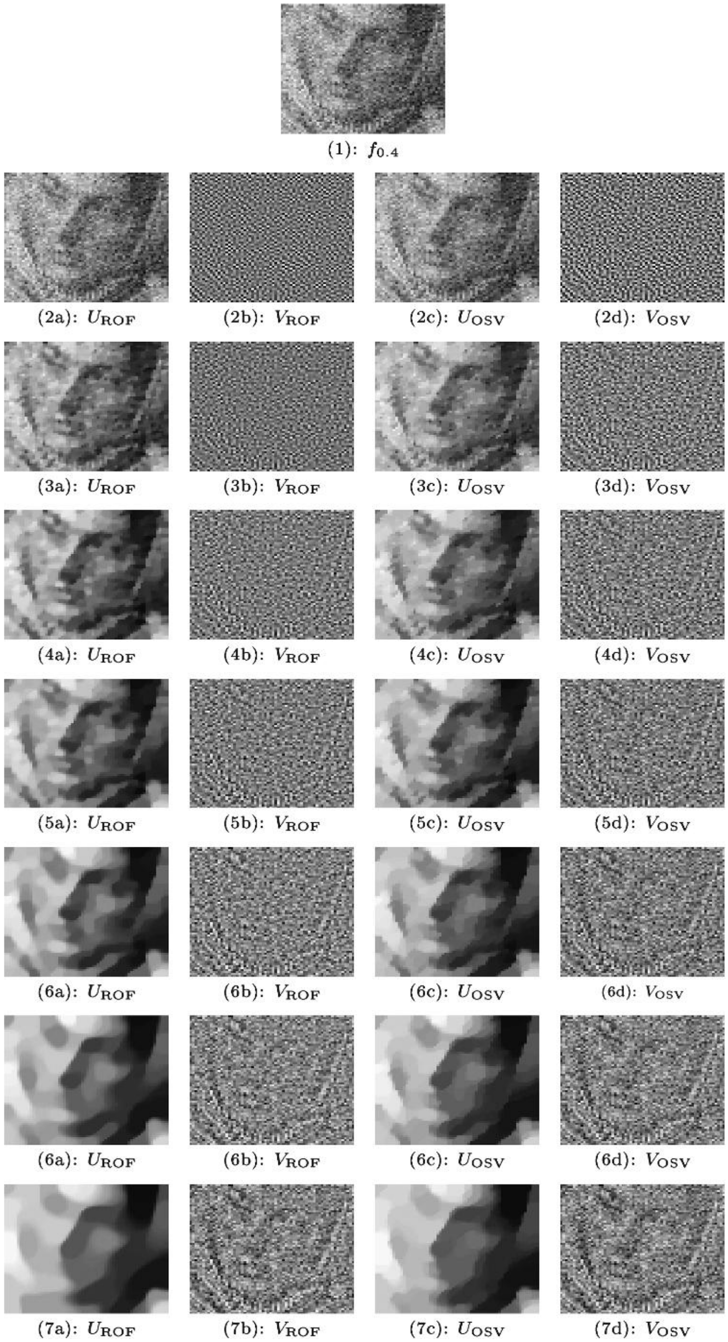

FIG. 4. The decomposition of the Barbara image degraded by additive Gaussian noise of mean 0 and standard deviation $\sigma = 0.4$.

Hence,

$$\|\hat{u}_{\epsilon,h} - U_{\epsilon,h}\|_{L^2(0,s;\mathcal{F}^h)} \leqslant \frac{\Delta t}{\sqrt{3}}\sqrt{\hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(0))} \quad \forall s \in [0,T].$$

Inequalities (4.4) and (3.10) give that, for all $s \in [0,T]$,

$$\begin{aligned}\|u'_{\epsilon,h} - U'_{\epsilon,h}\|_{L^2(0,s;\mathcal{F}^h)} &\leqslant \|U'_{\epsilon,h}\|_{L^2(0,s;\mathcal{F}^h)} + \|U'_{\epsilon,h}\|_{L^2(0,s;\mathcal{F}^h)} \\ &\leqslant \sqrt{\hat{J}_{\lambda,\epsilon,h}(u_{\epsilon,h}(0))} + \sqrt{\hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(0))}.\end{aligned}$$

It follows that

$$\|u_{\epsilon,h}(s) - U_{\epsilon,h}(s)\|^2_{-h,h} \leqslant \|u_{\epsilon,h}(0) - U_{\epsilon,h}(0)\|^2_{-h,h} + C^2\Delta t \quad \forall s \in [0,T],$$

TABLE 1 *ROF data for Figs 1–4*

$\|N_\sigma\|$	Figure	λ_{ROF}	Δt_{ROF}	N_{ROF}	T_{ROF}	$\|V_{\mathrm{ROF}}\|$
0	1 (2a), (2b)	13260	2.2016×10^{-06}	980	2.1576×10^{-03}	0.010005
0	1 (3a), (3b)	3808	6.6928×10^{-06}	990	6.6259×10^{-03}	0.0300124
0	1 (4a), (4b)	1950	1.1214×10^{-05}	980	1.0990×10^{-02}	0.0500392
0	1 (5a), (5b)	1151	1.7248×10^{-05}	970	1.6731×10^{-02}	0.0700415
0	1 (6a), (6b)	720	2.1920×10^{-05}	980	2.1482×10^{-02}	0.0900083
0	1 (7a), (7b)	465	2.6417×10^{-05}	990	2.6152×10^{-02}	0.110094
0	1 (8a), (8b)	314	3.5568×10^{-05}	950	3.3790×10^{-02}	0.13012
0.0353264	2 (2a), (2b)	48300	6.4800×10^{-07}	960	6.2208×10^{-04}	0.00352973
0.0353264	2 (3a), (3b)	8250	3.3488×10^{-06}	990	3.3153×10^{-03}	0.0176771
0.0353264	2 (4a), (4b)	3450	7.2345×10^{-06}	990	7.1622×10^{-03}	0.035353
0.0353264	2 (5a), (5b)	1960	1.1146×10^{-05}	980	1.0923×10^{-02}	0.052944
0.0353264	2 (6a), (6b)	1232	1.6095×10^{-05}	970	1.5612×10^{-02}	0.0706694
0.0353264	2 (7a), (7b)	549	2.4365×10^{-05}	990	2.4121×10^{-02}	0.106003
0.0353264	2 (8a), (8b)	271	3.9254×10^{-05}	970	3.8077×10^{-02}	0.141388
0.141306	3 (2a), (2b)	13900	2.2010×10^{-06}	950	2.0910×10^{-03}	0.0141236
0.141306	3 (3a), (3b)	2170	1.0356×10^{-05}	990	1.0253×10^{-02}	0.0707037
0.141306	3 (4a), (4b)	1140	1.6949×10^{-05}	980	1.6610×10^{-02}	0.105957
0.141306	3 (5a), (5b)	597	2.4710×10^{-05}	980	2.4216×10^{-02}	0.141348
0.141306	3 (6a), (6b)	299	3.6128×10^{-05}	980	3.5406×10^{-02}	0.176658
0.141306	3 (7a), (7b)	167	6.0264×10^{-05}	960	5.7853×10^{-02}	0.211773
0.141306	3 (8a), (8b)	105	9.1476×10^{-05}	950	8.6902×10^{-02}	0.247464
0.282611	4 (2a), (2b)	7180	4.1002×10^{-06}	980	4.0182×10^{-03}	0.0282333
0.282611	4 (3a), (3b)	846	2.3285×10^{-05}	990	2.3052×10^{-02}	0.169611
0.282611	4 (4a), (4b)	480	3.2612×10^{-05}	980	3.1960×10^{-02}	0.226231
0.282611	4 (5a), (5b)	339	3.7462×10^{-05}	970	3.6338×10^{-02}	0.254513
0.282611	4 (6a), (6b)	226	4.8384×10^{-05}	950	4.5965×10^{-02}	0.282409
0.282611	4 (7a), (7b)	148.1	6.6576×10^{-05}	950	6.3247×10^{-02}	0.310933
0.282611	4 (8a), (8b)	99.2	9.2030×10^{-05}	970	8.9269×10^{-02}	0.339216

TABLE 2 *OSV data for Figs 1–4*

$\|N_\sigma\|$	Figure	λ_{OSV}	Δt_{OSV}	N_{OSV}	T_{OSV}	$\|V_{OSV}\|$
0	1 (2c), (2d)	$2.8 \times 10^{+08}$	1.0880×10^{-10}	950	1.0336×10^{-07}	0.0100051
0	1 (3c), (3d)	$7.29 \times 10^{+07}$	3.2760×10^{-10}	990	3.2432×10^{-07}	0.0299864
0	1 (4c), (4d)	$3.07 \times 10^{+07}$	6.7680×10^{-10}	990	6.7003×10^{-07}	0.0500044
0	1 (5c), (5d)	$1.361 \times 10^{+07}$	1.2497×10^{-09}	980	1.2247×10^{-06}	0.0700128
0	1 (6c), (6d)	$5.31 \times 10^{+06}$	2.2723×10^{-09}	960	2.1814×10^{-06}	0.0900054
0	1 (7c), (7d)	$2 \times 10^{+06}$	5.1941×10^{-09}	960	4.9864×10^{-06}	0.110038
0	1 (8c), (8d)	$8.14 \times 10^{+05}$	1.0881×10^{-08}	980	1.0663×10^{-05}	0.130008
0.0353264	2 (2c), (2d)	$1.073 \times 10^{+09}$	2.9670×10^{-11}	950	2.8186×10^{-08}	0.00353404
0.0353264	2 (3c), (3d)	$1.686 \times 10^{+08}$	1.6445×10^{-10}	990	1.6281×10^{-07}	0.0176724
0.0353264	2 (4c), (4d)	$6.41 \times 10^{+07}$	3.6400×10^{-10}	980	3.5672×10^{-07}	0.0353525
0.0353264	2 (5c), (5d)	$3.09 \times 10^{+07}$	6.7284×10^{-10}	980	6.5938×10^{-07}	0.053001
0.0353264	2 (6c), (6d)	$1.54 \times 10^{+07}$	1.1760×10^{-09}	970	1.1407×10^{-06}	0.0706285
0.0353264	2 (7c), (7d)	$3 \times 10^{+06}$	3.5391×10^{-09}	960	3.3976×10^{-06}	0.106037
0.0353264	2 (8c), (8d)	$5.93 \times 10^{+05}$	1.4820×10^{-08}	960	1.4228×10^{-05}	0.141306
0.141306	3 (2c), (2d)	$3.08 \times 10^{+08}$	9.9000×10^{-11}	950	9.4050×10^{-08}	0.0141382
0.141306	3 (3c), (3d)	$3.81 \times 10^{+07}$	5.5366×10^{-10}	990	5.4812×10^{-07}	0.0706924
0.141306	3 (4c), (4d)	$1.55 \times 10^{+07}$	1.1232×10^{-09}	990	1.1120×10^{-06}	0.106012
0.141306	3 (5c), (5d)	$4.88 \times 10^{+06}$	2.3069×10^{-09}	980	2.2607×10^{-06}	0.141339
0.141306	3 (6c), (6d)	$9.99 \times 10^{+05}$	8.7382×10^{-09}	970	8.4761×10^{-06}	0.176663
0.141306	3 (7c), (7d)	$2.31 \times 10^{+05}$	3.5756×10^{-08}	970	3.4683×10^{-05}	0.212065
0.141306	3 (8c), (8d)	$8.23 \times 10^{+04}$	1.0598×10^{-07}	970	1.0280×10^{-04}	0.247397
0.282611	4 (2c), (2d)	$1.574 \times 10^{+08}$	1.8302×10^{-10}	990	1.8119×10^{-07}	0.0282876
0.282611	4 (3c), (3d)	$1.252 \times 10^{+07}$	1.4240×10^{-09}	990	1.4098×10^{-06}	0.169625
0.282611	4 (4c), (4d)	$4.76 \times 10^{+06}$	2.5168×10^{-09}	980	2.4665×10^{-06}	0.226235
0.282611	4 (5c), (5d)	$2.24 \times 10^{+06}$	4.3344×10^{-09}	980	4.2477×10^{-06}	0.254409
0.282611	4 (6c), (6d)	$8.23 \times 10^{+05}$	9.9369×10^{-09}	980	9.7381×10^{-06}	0.282693
0.282611	4 (7c), (7d)	$2.48 \times 10^{+05}$	3.1450×10^{-08}	970	3.0506×10^{-05}	0.311107
0.282611	4 (8c), (8d)	$9.2 \times 10^{+04}$	8.5795×10^{-08}	980	8.4079×10^{-05}	0.33921

where $C \equiv C(U_{\epsilon,h}(0), u_{\epsilon,h}(0), \epsilon, \lambda, \Omega) > 0$ is defined by

$$C^2 = \frac{2}{\sqrt{3}}\sqrt{\hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(0))}\left(\sqrt{\hat{J}_{\lambda,\epsilon,h}(u_{\epsilon,h}(0))} + \sqrt{\hat{J}_{\lambda,\epsilon,h}(U_{\epsilon,h}(0))}\right).$$

Taking square roots and using the inequality $\sqrt{a^2+b^2} \leqslant |a| + |b|$ give the stated result. □

5. Numerical simulations

In this section, we present numerical results which illustrate the superiority of the OSV model over the second-order Rudin–Osher–Fatemi (ROF) model presented in Rudin *et al.* (1992).

In order that the term $U^n_{\epsilon,h} - f$ is treated consistently, we modify the scheme $(\mathbf{P}_1^{\epsilon,h,\Delta t})$ in (3.1) thus

$$(\hat{\mathcal{G}}^h d_t U^n_{\epsilon,h}, \eta_h)^h + \left(\frac{\nabla U^n_{\epsilon,h}}{|\nabla U^{n-1}_{\epsilon,h}|_\epsilon}, \nabla \eta_h \right) = -\lambda (\mathcal{G}^h [U^n_{\epsilon,h} - f], \eta_h)^h \quad \forall \eta_h \in S^h. \tag{5.1}$$

For the ROF model, we use the analogous scheme

$$(d_t U^n_{\epsilon,h}, \eta_h)^h + \left(\frac{\nabla U^n_{\epsilon,h}}{|\nabla U^{n-1}_{\epsilon,h}|_\epsilon}, \nabla \eta_h \right) = -\lambda (U^n_{\epsilon,h} - f, \eta_h) \quad \forall \eta_h \in S^h. \tag{5.2}$$

We take $\Omega = (0,1) \times (0,1)$ and $\epsilon = 1 \times 10^{-5}$. The minimum and maximum of the computed image are coloured black and white. Thus, relative contrast, rather than absolute greyscale intensity, is depicted. For initial data f and solution $U(\cdot, t)$ at time t, $V(\cdot, t) := f - U(\cdot, t)$ is defined to be the part of the image removed (see Section 1).

The Barbara image f_0 (which is defined on an $N_h = 64 \times 64$ grid and has minimum value -1 and mean value 0) in Fig. 1(1) is considered. Noise is sampled from a Gaussian distribution with mean

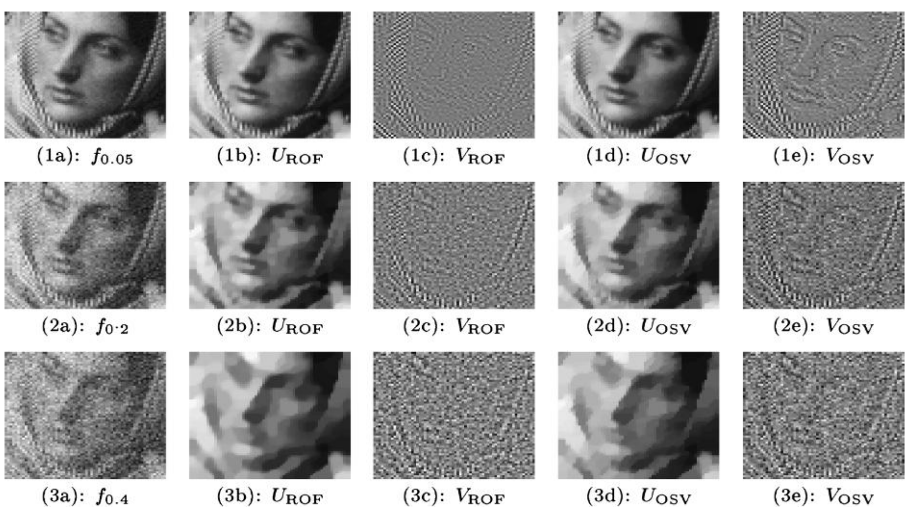

FIG. 5. The decomposition of the Barbara image degraded by additive Gaussian noise of mean 0 and standard deviation $\sigma \in \{0.05, 0.2, 0.4\}$.

TABLE 3 *Values of* $\|N_\sigma\|$, λ, Δt, N, T, $\|V\|$ *for Fig.* 5.

$\|N_\sigma\|$	Figure	λ	Δt	N	T	$\|V\|$
0.0353264	(1a), (1b)	3450	7.2345×10^{-06}	990	7.1622×10^{-03}	0.035353
	(1c), (1d)	$6.41 \times 10^{+07}$	3.6400×10^{-10}	980	3.5672×10^{-07}	0.0353525
0.141306	(2a), (2b)	597	2.4710×10^{-05}	980	2.4216×10^{-02}	0.141348
	(2c), (2d)	$4.88 \times 10^{+06}$	2.3069×10^{-09}	980	2.2607×10^{-06}	0.141339
0.282611	(3a), (3b)	226	4.8384×10^{-05}	950	4.5965×10^{-02}	0.282409
	(3c), (3d)	$8.23 \times 10^{+05}$	9.9369×10^{-09}	980	9.7381×10^{-06}	0.282693

0 and standard deviation σ (generated using the Matlab function ‘randn’). The mean of the noise is subtracted from it, giving a discrete function of mean 0 which is added to f_0 to give a noisy version f_σ. The decomposition of f_0, $f_{0.05}$, $f_{0.2}$ and $f_{0.4}$ is considered in Figs 1–4. For various values of λ_{ROF}, the discrete steady states U_{ROF} and V_{ROF} for the ROF model are depicted in (a) and (b), and for various values of λ_{OSV}, the discrete steady states U_{OSV} and V_{OSV} for the OSV model are depicted in (c) and (d). In each row, the values of λ_{ROF} and λ_{OSV} are chosen such that $\|V_{\text{ROF}}\| \approx \|V_{\text{OSV}}\|$. In each simulation, Δt is chosen such that approximately 1000 time steps are taken to reach steady state, which is assumed to have been achieved when

$$\frac{\|V^n - V^{n-1}\|}{\|V^{n-1}\|} \leqslant 1 \times 10^{-6}.$$

Values of λ_{ROF}, Δt_{ROF}, N_{ROF}, T_{ROF}, $\|V_{\text{ROF}}\|$, λ_{OSV}, Δt_{OSV}, N_{OSV}, T_{OSV} and $\|V_{\text{OSV}}\|$ are given in Tables 1 and 2.

It is seen that if $\|V_{\text{ROF}}\| \approx \|V_{\text{OSV}}\|$, then V_{ROF} contains more structure than V_{OSV}. Hence, the OSV model is better at recovering the fine detail present in an image.

The images $f_{0.05}$, $f_{0.2}$ and $f_{0.4}$ are considered again in Fig. 5 (see (1a)–(3a)). In each row, λ_{ROF} and λ_{OSV} are chosen such that $\|V_{\text{ROF}}\|, \|V_{\text{OSV}}\| \approx \|N_\sigma\|$, where N_σ is the discrete function which is added to f_0 to give f_σ. The functions U_{ROF}, V_{ROF}, U_{OSV} and V_{OSV} are depicted in (b), (c), (d) and (e).

Values of $\|N_\sigma\|$, λ_{ROF}, Δt_{ROF}, N_{ROF}, T_{ROF}, $\|V_{\text{ROF}}\|$, λ_{OSV}, Δt_{OSV}, N_{OSV}, T_{OSV} and $\|V_{\text{OSV}}\|$ are given in Table 3.

It is seen that for each value of σ, V_{ROF} and V_{OSV} both contain some structure, but the former much more so than the latter.

6. Conclusions

The analysis of the OSV model carried out by Elliott & Smitheman (2007) has been continued. Fully and linearly implicit finite-element approximations to a regularized version of the OSV IBVP and a fully implicit approximation to a weak formulation of the original problem have been introduced. Well-posedness and unconditional Lyapunov stability of the fully discrete schemes have been proved. Convergence results as the spatial mesh parameter, time step size and the regularization parameter tend to 0 have been proved. Suboptimal rates of convergence as the time step size and the regularization parameter tend to 0 have been found. Numerical results illustrating the superiority of the OSV model over the second-order ROF model have been presented.

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Appendix: Proof of Lemma 2.4

(i) Inequalities (2.4) and (2.6–2.9) give that

$$\begin{aligned}\forall\,\eta_h \in V^h, \quad \left|\|\eta_h\|^2_{-h,h} - \|\eta_h\|^2_{-h}\right| &= |(\eta_h, \hat{\mathcal{G}}^h\eta_h)^h - (\eta_h, \mathcal{G}^h\eta_h)| \\ &\leqslant |(\eta_h, \hat{\mathcal{G}}^h\eta_h)^h - (\eta_h, \hat{\mathcal{G}}^h\eta_h)| + |(\eta_h, [\hat{\mathcal{G}}^h - \mathcal{G}^h]\eta_h)| \\ &\leqslant Ch\|\eta_h\|_{-h,h}\|\eta_h\| + \|(\hat{\mathcal{G}}^h - \mathcal{G}^h)\eta_h\|\|\eta_h\| \leqslant Ch\|\eta_h\|^2.\end{aligned}$$

Inequality (2.11) gives that

$$\forall\,\eta_h \in V^h, \quad \left|\|\eta_h\|^2_{-h} - \|\eta_h\|^2_{-1}\right| = |(\eta_h, [\mathcal{G}^h - \mathcal{G}]\eta_h)| \leqslant \|[\mathcal{G}^h - \mathcal{G}]\eta_h\|\|\eta_h\| \leqslant Ch\|\eta_h\|^2.$$

(ii) The proof of (i) gives that for all $\eta_h \in L^2(0, T; V^h)$ and $s \in [0, T]$,

$$\begin{aligned}\left|\|\eta_h\|^2_{L^2(0,s;\mathcal{F}^h)} - \|\eta_h\|^2_{L^2(0,s;(\mathcal{F},\|\cdot\|_{-h}))}\right| &= \left|\int_0^s [\|\eta_h(t)\|^2_{-h,h} - \|\eta_h(t)\|^2_{-h}]\mathrm{d}t\right| \\ &\leqslant Ch\int_0^s \|\eta_h(t)\|^2\mathrm{d}t = Ch\|\eta_h\|^2_{L^2(0,s;L^2(\Omega))},\end{aligned}$$

and, similarly,

$$\left|\|\eta_h\|^2_{L^2(0,s;(\mathcal{F},\|\cdot\|_{-h}))} - \|\eta_h\|^2_{L^2(0,s;\mathcal{F})}\right| \leqslant Ch\|\eta_h\|^2_{L^2(0,s;L^2(\Omega))}.$$

(iii) Note that

$$\|\nabla[\hat{\mathcal{G}}^h\eta_h - \mathcal{G}\eta]\|^2_{L^2(\Omega_T)} = \|\eta_h\|^2_{L^2(0,T;\mathcal{F}^h)} + \|\eta\|^2_{L^2(0,T;\mathcal{F})} - 2\int_0^T (\nabla\hat{\mathcal{G}}^h\eta_h(t), \nabla\mathcal{G}\eta(t))\mathrm{d}t.$$

The Poincaré inequality (2.3) and inequality (2.12) give that

$$\begin{aligned}
&\left|\int_0^T (\nabla\hat{\mathcal{G}}^h\eta_h(t), \nabla\mathcal{G}\eta(t))\mathrm{d}t - \|\eta\|^2_{L^2(0,T;\mathcal{F})}\right| \\
&\quad = \left|\int_0^T (\hat{\mathcal{G}}^h\eta_h(t) - \mathcal{G}\eta(t), \eta(t))\mathrm{d}t\right| \\
&\quad \leqslant \|\eta\|_{L^2(\Omega_T)}\|\hat{\mathcal{G}}^h\eta_h - \mathcal{G}\eta\|_{L^2(\Omega_T)} \\
&\quad \leqslant \|\eta\|_{L^2(\Omega_T)}\left(\|(\hat{\mathcal{G}}^h - \mathcal{G})\eta_h\|_{L^2(\Omega_T)} + \|\mathcal{G}(\eta_h - \eta)\|_{L^2(\Omega_T)}\right) \\
&\quad \leqslant \|\eta\|_{L^2(\Omega_T)}\left(Ch\|\eta_h\|_{L^2(\Omega_T)} + C_P\|\eta_h - \eta\|_{L^2(0,T;\mathcal{F})}\right) \to 0 \quad \text{as } h \downarrow 0.
\end{aligned}$$

Part (ii) gives that

$$\lim_{h\downarrow 0}\|\nabla[\hat{\mathcal{G}}^h\eta_h - \mathcal{G}\eta]\|^2_{L^2(\Omega_T)} = \|\eta\|^2_{L^2(0,T;\mathcal{F})} + \|\eta\|^2_{L^2(0,T;\mathcal{F})} - 2\|\eta\|^2_{L^2(0,T;\mathcal{F})} = 0.$$

(iv) The proof of (ii) and inequality (2.12) give that for all $\eta_h, \xi_h \in L^2(0, T; V^h)$ and $s \in [0, T]$,

$$\begin{aligned}
&|\|\nabla[\hat{\mathcal{G}}^h\eta_h - \mathcal{G}^h\xi_h]\|^2_{L^2(0,s;L^2(\Omega))} - \|\eta_h - \xi_h\|^2_{L^2(0,s;\mathcal{F})}| \\
&= \left|\int_0^s \Big[\|\eta_h(t)\|^2_{-h,h} - \|\eta_h(t)\|^2_{-1} + \|\xi_h(t)\|^2_{-h} - \|\xi_h(t)\|^2_{-1}\right. \\
&\qquad\left. + 2\{(\nabla\mathcal{G}\eta_h(t), \nabla\mathcal{G}\xi_h(t)) - (\nabla\hat{\mathcal{G}}^h\eta_h(t), \nabla\mathcal{G}^h\xi_h(t))\}\Big]\,\mathrm{d}t\right| \\
&= \Big|\|\eta_h\|^2_{L^2(0,s;\mathcal{F}^h)} - \|\eta_h\|^2_{L^2(0,s;\mathcal{F})} + \|\xi_h\|^2_{L^2(0,s;(\mathcal{F},\|\cdot\|_{-h}))} - \|\xi_h\|^2_{L^2(0,s;\mathcal{F})} \\
&\qquad + 2\int_0^s([\mathcal{G} - \hat{\mathcal{G}}^h]\eta_h(t), \xi_h(t))\mathrm{d}t\Big| \\
&\leqslant \left|\|\eta_h\|^2_{L^2(0,s;\mathcal{F}^h)} - \|\eta_h\|^2_{L^2(0,s;\mathcal{F})}\right| + \left|\|\xi_h\|^2_{L^2(0,s;(\mathcal{F},\|\cdot\|_{-h}))} - \|\xi_h\|^2_{L^2(0,s;\mathcal{F})}\right| \\
&\qquad + 2\|(\mathcal{G} - \hat{\mathcal{G}}^h)\eta_h\|_{L^2(0,s;L^2(\Omega))}\|\xi_h\|_{L^2(0,s;L^2(\Omega))} \\
&\leqslant Ch[\|\eta_h\|_{L^2(0,s;L^2(\Omega))} + \|\xi_h\|_{L^2(0,s;L^2(\Omega))}]^2.
\end{aligned}$$