IMA Journal of Numerical Analysis Advance Access published September 22, 2012

IMA Journal of Numerical Analysis (2012) Page 1 of 26
doi:10.1093/imanum/drs022

Finite element analysis for a coupled bulk–surface partial differential equation

CHARLES M. ELLIOTT AND THOMAS RANNER*
Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
*Corresponding author: t.ranner@warwick.ac.uk

[Received on 17 January 2012; revised on 28 June 2012]

In this paper, we define a new finite element method for numerically approximating the solution of a partial differential equation in a bulk region coupled with a surface partial differential equation posed on the boundary of the bulk domain. The key idea is to take a polyhedral approximation of the bulk region consisting of a union of simplices, and to use piecewise polynomial boundary faces as an approximation of the surface. Two finite element spaces are defined, one in the bulk region and one on the surface, by taking the set of all continuous functions which are also piecewise polynomial on each bulk simplex or boundary face. We study this method in the context of a model elliptic problem; in particular, we look at well-posedness of the system using a variational formulation, derive perturbation estimates arising from domain approximation and apply these to find the optimal-order error estimates. A numerical experiment is described which demonstrates the order of convergence.

Keywords: surface finite elements; error analysis; bulk–surface elliptic equations.

1. Introduction

Coupled bulk–surface partial differential equations arise in many applications; for example, they arise naturally in fluid dynamics and biological applications. This paper studies mathematically a finite element approach to a sample elliptic problem. The method is based on taking a polyhedral approximation of the domain. Given a sufficiently smooth boundary, we go on to show error bounds of order h^k in the H^1 norm and order h^{k+1} in the L^2 norm, where k is the polynomial degree in the underlying finite element space.

1.1 *The coupled system*

For a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) with boundary Γ, we seek solutions $u\colon \Omega \to \mathbb{R}$ and $v\colon \Gamma \to \mathbb{R}$ of the system

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$(\alpha u - \beta v) + \frac{\partial u}{\partial \mathrm{n}} = 0 \quad \text{on } \Gamma, \tag{1.1b}$$

$$-\Delta_\Gamma v + v + \frac{\partial u}{\partial \mathrm{n}} = g \quad \text{on } \Gamma. \tag{1.1c}$$

Here we assume that α and β are given positive constants and that f and g are known functions on Ω and Γ, respectively. We denote by Δ_Γ the Laplace–Beltrami operator on Γ and by n the outward pointing normal to Γ.

1.2 *Applications*

In recent times there has been a great deal of attention paid to problems involving diffusion on a surface, for example, Dziuk & Elliott (2007b) and references therein. Of particular interest is cell biology; see, for example, Schwartz *et al.* (2005) and Sbalzarini *et al.* (2006). Indeed, cellular metabolism and signalling are mediated in part by trans-membrane receptors that can diffuse in the cell membrane; see Alberta *et al.* (2002). There are also examples where this surface diffusion is coupled with diffusion in the bulk. For example, fluorescence loss in photobleaching where surface diffusion of a signalling molecule, G-protein Rac, cycles between the cytoplasm (bulk) and cell membrane (surface); see Novak *et al.* (2007).

The coupling on the surface (1.1b, 1.1c) has been used by Novak *et al.* (2007). It can be viewed as a linearization of the more general equation

$$\frac{\partial u}{\partial \mathrm{n}} + L(u, v) = 0,$$

where $L_u > 0$ and $L_v < 0$, which has been used in Kwon & Derby (2001), Booty & Siegel (2010), Medvedev & Stuchebrukhov (2011) and Rätz & Röger (2011) for example. We leave the numerical analysis of more general couplings, the parabolic case and evolving domains, to future work.

1.3 *Outline of paper*

The paper is laid out as follows. In the second section, we will derive a variational form for the equations. The third section looks at existence, uniqueness and regularity of variational solutions. The fourth section focuses on the approximations we make to the geometry of the problem. In the fifth section, we develop the finite element method and in the sixth section we will look for error bounds for this method. In the final section, we will show some numerical results.

2. Derivation of variational form

2.1 *Surface properties*

Throughout we will use the notation from Deckelnick *et al.* (2005). We will assume that Γ is a compact $(N-1)$-dimensional hypersurface without boundary and that Γ is C^2, so there exists a distance function $d\colon \mathbb{R}^N \to \mathbb{R}$ defined by

$$d(x) = \begin{cases} -\inf\{|x-y| : y \in \Gamma\} & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Gamma, \\ \inf\{|x-y| : y \in \Gamma\} & \text{if } x \notin \overline{\Omega}. \end{cases}$$

Since $|\nabla d(x)| \equiv 1$ in a neighbourhood about Γ, we can define the normal to Γ for almost every $x \in \Gamma$ by

$$\mathrm{n}(x) = \nabla d(x).$$

It follows that there exists a narrow band $U = \{x \in \mathbb{R}^N : |d(x)| < \delta_\Gamma\}$ about Γ, such that $d \in C^2(U)$, for which we can also define the normal projection $x \mapsto p(x)$ from U onto Γ given by the unique solution of

$$x = p(x) + d(x)\mathrm{n}(p(x)). \tag{2.1}$$

This is possible by the assumptions above; see, for example, Hildebrant (1982). Note that $p(x)$ is the closest point to x on Γ, so p is also the closest point operator. Since this decomposition is unique, we can extend n to a vector field on all of U so that $\mathrm{n}(x) = \mathrm{n}(p(x))$.

For a function $\xi : \Gamma \to \mathbb{R}$, we define its surface gradient to be

$$\nabla_\Gamma \xi := \nabla\xi - (\nabla\xi \cdot \mathrm{n})\mathrm{n},$$

where $\nabla\xi$ denotes the gradient with respect to ambient coordinates of an arbitrary extension to U of ξ. Alternatively, we can denote this relation as $\nabla_\Gamma \xi = P\nabla\xi$, where P is an $N \times N$ tensor given by $P_{ij} = \delta_{ij} - \mathrm{n}_i\mathrm{n}_j$. Note that P is symmetric. The Laplace–Beltrami operator is given by the surface divergence of the surface gradient, that is,

$$\Delta_\Gamma \xi := \nabla_\Gamma \cdot \nabla_\Gamma \xi.$$

We denote by $\mathcal{H} = \nabla_\Gamma \cdot \mathrm{n}$ the mean curvature of Γ. For facts about tangential gradients, see Gilbarg & Trudinger (1983, Chapter 16).

We denote by $\mathrm{d}o$ the $(N-1)$-dimensional surface measure on Γ. The formula for integration by parts on Γ is given by

$$\int_\Gamma (\nabla_\Gamma)_i \xi \,\mathrm{d}o = -\int_\Gamma \xi \mathcal{H} \mathrm{n}_i \,\mathrm{d}o.$$

This gives us a surface Green's formula for a surface without boundary,

$$\int_\Gamma (-\Delta_\Gamma y)\xi \,\mathrm{d}o = \int_\Gamma \nabla_\Gamma y \cdot \nabla_\Gamma \xi \,\mathrm{d}o. \tag{2.2}$$

2.2 *Variational form*

We take functions η, ξ in a suitable space of test functions, multiply (1.1a) by η and (1.1c) by ξ, and integrate by parts to get

$$\int_\Omega \nabla u \cdot \nabla\eta + u\eta \,\mathrm{d}x - \int_\Gamma \eta \frac{\partial u}{\partial \mathrm{n}} \,\mathrm{d}o = \int_\Omega f\eta \,\mathrm{d}x, \tag{2.3a}$$

$$\int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \xi + v\xi \,\mathrm{d}o + \int_\Gamma \frac{\partial u}{\partial \mathrm{n}} \xi \,\mathrm{d}o = \int_\Gamma g\xi \,\mathrm{d}o. \tag{2.3b}$$

The boundary condition (1.1b) gives us that

$$-\int_\Gamma \eta \frac{\partial u}{\partial \mathrm{n}} \,\mathrm{d}o = \int_\Gamma (\alpha u - \beta v)\eta \,\mathrm{d}o \quad \text{and} \quad \int_\Gamma \frac{\partial u}{\partial \mathrm{n}} \xi \,\mathrm{d}o = -\int_\Gamma (\alpha u - \beta v)\xi \,\mathrm{d}o. \tag{2.4}$$

We substitute these into (2.3) to get

$$\int_\Omega \nabla u \cdot \nabla\eta + u\eta \,\mathrm{d}x + \int_\Gamma (\alpha u - \beta v)\eta \,\mathrm{d}o = \int_\Omega f\eta \,\mathrm{d}x, \tag{2.5a}$$

$$\int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \xi + v\xi \,\mathrm{d}o - \int_\Gamma (\alpha u - \beta v)\xi \,\mathrm{d}o = \int_\Gamma g\eta \,\mathrm{d}o. \tag{2.5b}$$

We now take a weighted sum of (2.5a) and (2.5b) to obtain the variational form

$$\alpha \int_\Omega (\nabla u \cdot \nabla \eta + u\eta)\ \mathrm{d}x + \beta \int_\Gamma (\nabla_\Gamma v \cdot \nabla_\Gamma \xi + v\xi)\ \mathrm{d}o$$
$$+ \int_\Gamma (\alpha u - \beta v)(\alpha\eta - \beta\xi)\,\mathrm{d}o = \alpha \int_\Omega f\eta\,\mathrm{d}x + \beta \int_\Gamma g\xi\,\mathrm{d}o. \tag{2.6}$$

To help with notation later, we will write $a((u, v), (\eta, \xi))$ for the left-hand side of this equation and $l((\eta, \xi))$ for the right-hand side.

We will test this variational form over the space $H^1(\Omega) \times H^1(\Gamma)$ which we define to be

$$H^1(\Omega) \times H^1(\Gamma) := \{(\eta, \xi) \mid \eta \in H^1(\Omega), \xi \in H^1(\Gamma)\}. \tag{2.7}$$

We equip this space with the inner product

$$\langle(\eta_1, \xi_1), (\eta_2, \xi_2)\rangle_{H^1(\Omega)\times H^1(\Gamma)} = \langle\eta_1, \eta_2\rangle_{H^1(\Omega)} + \langle\xi_1, \xi_2\rangle_{H^1(\Gamma)}, \tag{2.8}$$

and induced norm given by

$$\|(\eta, \xi)\|_{H^1(\Omega)\times H^1(\Gamma)} = (\|\eta\|^2_{H^1(\Omega)} + \|\xi\|^2_{H^1(\Gamma)})^{1/2}. \tag{2.9}$$

One may define higher-order spaces if Γ is more regular: to define $H^l(\Omega) \times H^l(\Gamma)$, we require Γ to be $C^{j,\kappa}$ with $l \leqslant j + \kappa$ and $\kappa = 0, 1$. See Wloka (1987) for details of how to define the surface Sobolev spaces.

Hence the variational formulation of the problem is to find $(u, v) \in H^1(\Omega) \times H^1(\Gamma)$ such that

$$a((u, v), (\eta, \xi)) = l((\eta, \xi)) \quad \text{for all } (\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma). \tag{2.10}$$

3. Existence, uniqueness and regularity

In this section, we apply the usual Lax–Milgram techniques (Evans, 1998) to the variational form developed in Section 2 in order to find a unique solution to (2.10). Following this, we split the equations to show regularity with respect to the bulk and surface variables independently. To apply these techniques we must show that a is bounded and coercive and l is bounded over $H^1(\Omega) \times H^1(\Gamma)$.

To see that a is bounded, note that

$$\begin{aligned} a((w, y), (\eta, \xi)) &\leqslant \alpha\|w\|_{H^1(\Omega)}\|\eta\|_{H^1(\Omega)} + \beta\|y\|_{H^1(\Gamma)}\|\xi\|_{H^1(\Gamma)} \\ &\quad + \int_\Gamma (\alpha w - \beta y)(\alpha\eta - \beta\xi)\,\mathrm{d}o \\ &\leqslant \sqrt{2}\max\{\alpha, \beta\}\|(w, y)\|_{H^1(\Omega)\times H^1(\Gamma)}\|(\eta, \xi)\|_{H^1(\Omega)\times H^1(\Gamma)} \\ &\quad + 2c_T^2 \max\{\alpha, \beta\}^2\|(w, y)\|_{H^1(\Omega)\times H^1(\Gamma)}\|(\eta, \xi)\|_{H^1(\Omega)\times H^1(\Gamma)} \\ &\leqslant c\|(w, y)\|_{H^1(\Omega)\times H^1(\Gamma)}\|(\eta, \xi)\|_{H^1(\Omega)\times H^1(\Gamma)}. \end{aligned} \tag{3.1}$$

Here, c_T is the constant from the trace theorem; see Evans (1998). Coercivity of a is immediate since we have

$$a((\eta,\xi),(\eta,\xi)) = \alpha\|\eta\|^2_{H^1(\Omega)} + \beta\|\xi\|^2_{H^1(\Gamma)} + \|\alpha\eta - \beta\xi\|^2_{L^2(\Gamma)}$$
$$\geqslant \sqrt{2}\min\{\alpha,\beta\}\|(\eta,\xi)\|^2_{H^1(\Omega)\times H^1(\Gamma)}. \tag{3.2}$$

Hence a is coercive if $\alpha,\beta > 0$. By the Cauchy–Schwarz inequality, l is clearly bounded.

THEOREM 3.1 (Existence and uniqueness) Given $f \in H^{-1}(\Omega)$, $g \in H^{-1}(\Gamma)$ and $\alpha,\beta > 0$, there exists a unique pair $(u,v) \in H^1(\Omega) \times H^1(\Gamma)$ such that

$$a((u,v),(\eta,\xi)) = l((\eta,\xi)) \quad \text{for all } (\eta,\xi) \in H^1(\Omega)\times H^1(\Gamma). \tag{3.3}$$

Furthermore, if Γ is C^3, we can achieve bounds in the H^2 norms by setting η and ξ equal to zero in turn.

For $\eta = 0$, we get

$$\beta\int_\Gamma \nabla_\Gamma v\cdot\nabla_\Gamma\xi + v\xi\,\mathrm{d}o + \int_\Gamma \beta^2 v\xi\,\mathrm{d}o = \beta\int_\Gamma g\xi\,\mathrm{d}o + \int_\Gamma \alpha\beta u\xi\,\mathrm{d}o. \tag{3.4}$$

This is exactly the variational form of the equation

$$-\beta\Delta_\Gamma v + (\beta+\beta^2)v = \beta g + \alpha\beta u \quad \text{on } \Gamma. \tag{3.5}$$

Hence by surface elliptic theory (Aubin, 1982), if Γ is C^3, we have that $v \in H^2(\Gamma)$. Since, by the trace theorem, $u \in L^2(\Gamma)$, we have the bound

$$\|v\|_{H^2(\Gamma)} \leqslant c(\|g\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)} + \|u\|_{H^1(\Omega)}). \tag{3.6}$$

For $\xi = 0$, we get

$$\alpha\int_\Omega \nabla u\cdot\nabla\eta + u\eta\,\mathrm{d}x + \int_\Gamma \alpha^2 u\eta\,\mathrm{d}o = \alpha\int_\Omega f\eta\,\mathrm{d}x + \int_\Gamma \alpha\beta v\eta\,\mathrm{d}o. \tag{3.7}$$

This equation arises as the variational form of the equations

$$-\alpha\Delta u + \alpha u = \alpha f \quad \text{in } \Omega, \tag{3.8a}$$
$$\frac{\partial u}{\partial \mathrm{n}} + \alpha u = \beta v \quad \text{on } \Gamma. \tag{3.8b}$$

By the regularity theory of elliptic problems with Robin boundary data (see Ladyzhenskaia & Uraltseva, 1968; Gilbarg & Trudinger, 1983), if Γ is C^3, we have the following result:

$$\|u\|_{H^2(\Omega)} \leqslant c(\|f\|_{L^2(\Omega)} + \|v\|_{H^{1/2}(\Gamma)}). \tag{3.9}$$

THEOREM 3.2 (Regularity) If Γ is $C^3, f \in L^2(\Omega), g \in L^2(\Gamma)$ and $\alpha, \beta > 0$ and (u, v) solve the variational problem (2.6), then

$$u \in H^2(\Omega) \text{ and } v \in H^2(\Gamma),$$

and

$$\|(u, v)\|_{H^2(\Omega)\times H^2(\Gamma)} \leqslant c(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}). \tag{3.10}$$

4. Domain perturbation and estimates

4.1 *Domain approximation*

The first step we take in discretizing the system (1.1) is to take kth-order approximations $\Omega_h^{(k)}$ and $\Gamma_h^{(k)}$ of Ω and Γ. We follow ideas taken from Lenoir (1986), Bernardi (1989) and Dubois (1990) in order to define the triangulation of our bulk domain and use results of Dziuk (1988), Dziuk & Elliott (2007b) and Demlow (2009) to make estimates about the perturbation of the boundary of this domain. To prove the results in this section, we will assume Γ is C^{k+1}. The higher-order surface finite element spaces, used here, are described in Heine (2005).

Let $\check{\Omega}_h$ be a polyhedral approximation of Ω and $\check{\Gamma}_h = \partial\check{\Omega}_h$. We suppose that the faces of $\check{\Gamma}_h$ are $(N-1)$-simplices whose vertices lie on Γ so that $\check{\Gamma}_h$ is an interpolant of Γ. We take a quasi-uniform triangulation $\check{\mathscr{T}}_h$ of $\check{\Omega}_h$ (Brenner & Scott, 2002) consisting of closed simplices, either triangles in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$.

We define $h = \max\{\text{diam}(T) : T \in \check{\mathscr{T}}_h\}$ and assume that h is sufficiently small so that $\check{\Gamma}_h \subseteq U$, so that for all $x \in \check{\Gamma}_h$, there exists a unique point $p = p(x) \in \Gamma$ defined by (2.1). Finally, we assume that for each $T \in \check{\mathscr{T}}_h$, $T \cap \check{\Gamma}_h$ has at most one face of T.

4.1.1 *Exact triangulation.* In order to define our computational domains, we first define an exact triangulation of Ω. For each simplex $T \in \check{\mathscr{T}}_h$, we define an affine function $F_T : \mathbb{R}^N \to \mathbb{R}^N$ which maps the unit N-simplex $\hat{T}$ onto T (mapping the vertices of $\hat{T}$ onto the vertices of T) which we write as

$$F_T(\hat{x}) = A_T\hat{x} + b_T. \tag{4.1}$$

We say that a closed set T^{e} is a curved N-simplex if there exists a C^1 mapping F_T^{e} that maps $\hat{T}$ onto T^{e} that is of the form

$$F_T^{\mathrm{e}} = F_T + \Phi_T, \tag{4.2}$$

where F_T is the affine map from (4.1) and Φ_T is a C^1 mapping from $\hat{T}$ to $\mathbb{R}^N$ satisfying

$$C_T := \sup_{\hat{x}\in\hat{T}} |D\Phi_T(\hat{x})A_T^{-1}| \leqslant C < 1. \tag{4.3}$$

From this definition we immediately have the following results.

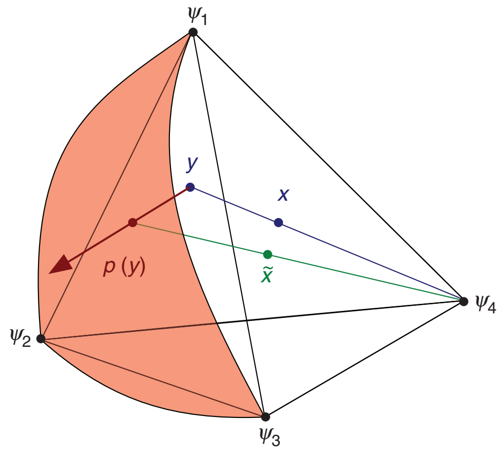

FIG. 1. An illustration of the construction of the exact triangulation of Ω. The point x is mapped onto $y \in \tau$ (the simplex spanned by ψ_1, ψ_2, ψ_3) and then to $\widetilde{x} = F_T^{\mathrm{e}}(x)$ by using the closest point projection (2.1) of y.

PROPOSITION 4.1 If the map F_T^{e} exists, then it is a C^1 diffeomorphism from $\hat{T}$ onto T^{e} and satisfies

$$\sup_{\hat{x}\in\hat{T}} |DF_T^{\mathrm{e}}(\hat{x})| \leqslant (1 + C_T)|A_T|,$$

$$\sup_{x\in T^{\mathrm{e}}} |D(F_T^{\mathrm{e}})^{-1}(x)| \leqslant (1 - C_T)^{-1}|A_T^{-1}|,$$

$$(1 - C_T)^N |\det A_T| \leqslant |\det DF_T^{\mathrm{e}}(\hat{x})| \leqslant (1 + C_T)^N |\det DF_T| \quad \text{for all } \hat{x} \in \hat{T}.$$

There are several ways of defining such a Φ_T given in the literature. Zlamal (1973, 1974) and Scott (1973) considered problems with finite element spaces defined over curved spaces. Scott gives an explicit construction of an exact triangulation in two dimensions which was generalized by Lenoir (1986). Here, in this paper, we use a construction based on Dubois (1990) which uses the normal projection (2.1). We will adopt the notation of Bänsch & Deckelnick (1999) and Deckelnick *et al.* (2009).

Bearing in mind our assumptions on the triangulation, each $T \in \check{\mathscr{T}}_h$ is either an internal simplex, with at most one node on $\check{\Gamma}_h$, in which case we set $\Phi_T = 0$; or T has more than one node on the boundary. We denote by l the number of nodes of T that lie in $\check{\Gamma}_h$ and denote by $\psi_1, \ldots, \psi_{N+1}$ the vertices of T, ordered so that $\psi_1, \ldots, \psi_l$ lie on $\check{\Gamma}_h$. For each point $x \in T$, we define barycentric coordinates by

$$x = \sum_{j=1}^{N+1} \lambda_j \psi_j$$

and write $\hat{x} = (\lambda_1, \ldots, \lambda_N)$ for the coordinates in $\hat{T}$. We next introduce

$$\lambda^* = \lambda^*(\hat{x}) = \sum_{j=1}^{l} \lambda_j, \quad \hat{\sigma} = \{\hat{x} \in \hat{T} : \lambda^*(\hat{x}) = 0\}.$$

In three dimensions, this falls into the following cases.

(1) $T \cap \check{\Gamma}_h$ is an edge of a tetrahedron ($l = 2$), then $\hat{\sigma}$ is the inverse image of the edge spanned by ψ_3, ψ_4 under F_T.

(2) $T \cap \check{\Gamma}_h$ is a face of a tetrahedron ($l = 3$), then $\hat{\sigma}$ is the point $F_T^{-1}(\psi_4)$.

For $\hat{x} \notin \hat{\sigma}$, we denote the projection of x onto τ by $y = y(\hat{x}) \in \tau$ by

$$y = \sum_{j=1}^{l} \frac{\lambda_j}{\lambda^*} \psi_j \in \tau.$$

Then using the normal projection $p(y) \in \Gamma$ of y given by (2.1) and we define Φ_T by (see Fig. 1)

$$\Phi_T(\hat{x}) = \begin{cases} (\lambda^*)^{k+2}(p(y) - y) & \text{if } \hat{x} \notin \hat{\sigma}, \\ 0 & \text{if } \hat{x} \in \hat{\sigma}. \end{cases} \tag{4.4}$$

We now follow a sequence of lemmas from Bernardi (1989) to show that Φ_T satisfies (4.3).

LEMMA 4.2 The mapping y is of class C^{k+1} on $\hat{T} \backslash \hat{\sigma}$ and satisfies

$$\|D_{\hat{x}}^m y\|_{L^\infty(\hat{T} \backslash \hat{\sigma})} \leqslant \frac{ch}{(\lambda^*)^m} \quad \text{for } 1 \leqslant m \leqslant k+1. \tag{4.5}$$

Proof. See Bernardi (1989, Lemma 6.3). □

LEMMA 4.3 The mapping $p(y)$ is of class C^{k+1} on $\hat{T} \backslash \hat{\sigma}$ and we have the bound

$$\|D_{\hat{x}}^m (p(y) - y)\|_{L^\infty(\hat{T} \backslash \hat{\sigma})} \leqslant \frac{ch^2}{(\lambda^*)^m}. \tag{4.6}$$

Proof. We remark, using Bernardi (1989, Equation 2.9),

$$\|D_{\hat{x}}^m (p(y) - y)\|_{L^\infty(\hat{T} \backslash \hat{\sigma})} \leqslant c \sum_{r=1}^{m} \left(\|D_y^r (p(y) - y)\|_{L^\infty(\tau)} \prod_{q=1}^{m} \|D_{\hat{x}}^q y\|_{L^\infty(\hat{T} \backslash \hat{\sigma})}^{i_q} \right)$$

where $\underline{i} = (i_1, \ldots, i_m)$ is a multiindex in $\mathbb{N}$.

$$\sum_{q=1}^{m} i_q = r \quad \text{and} \quad \sum_{q=1}^{m} q\, i_q = m.$$

We note that $p(y) = y$ if $y = \psi_j$ for any $0 \leqslant j \leqslant l$, so $y|_\tau$ can be seen as a linear interpolant of $p(y)$ on τ. Hence, from our geometric assumptions on Γ (Dziuk, 1988), $\|D_y^r (p(y) - y)\|_{L^\infty(\tau)} \leqslant ch^{2-r}$ for $0 \leqslant r \leqslant 2$.

Using (4.5) we see, if $m \leqslant 2$,

$$\|D_{\hat{x}}^m(p(y)-y)\|_{L^\infty(\hat{T}\setminus\hat{\sigma})} \leqslant c\sum_{r=1}^{m} h^{2-r}h^{(\sum_{q=1}^m i_q)}(\lambda^*)^{-(\sum_{q=1}^m q i_q)} \leqslant \frac{ch^2}{(\lambda^*)^m},$$

and if $m > 2$,

$$\|D_{\hat{x}}^m(p(y)-y)\|_{L^\infty(\hat{T}\setminus\hat{\sigma})} \leqslant c\left(\sum_{r=1}^{2} h^{2-r}h^{(\sum_{q=1}^m i_q)}(\lambda^*)^{-(\sum_{q=1}^m q i_q)} + \sum_{r=3}^{m} h^{(\sum_{q=1}^m i_q)}(\lambda^*)^{-(\sum_{q=1}^m q i_q)}\right) \leqslant \frac{ch^2}{(\lambda^*)^m}.$$

□

PROPOSITION 4.4 The mapping $\Phi_T(\hat{x}) = (\lambda^*)^{k+2}(p(y)-y)$ is of class C^{k+1} on $\hat{T}$ and satisfies

$$\|D^m\Phi_T\|_{L^\infty(\hat{T})} \leqslant ch^2 \quad \text{for } 0 \leqslant m \leqslant k+1. \tag{4.7}$$

Furthermore, Φ_T satisfies (4.3).

Proof. Using the Leibniz formula, we have for any $\hat{x}$ in $\hat{T}\setminus\hat{\sigma}$,

$$\begin{aligned} D_m\Phi_T(\hat{x}) &= D_{\hat{x}}^m((\lambda^*)^{k+2}(p(y)-y)) \\ &= \sum_{r=0}^{m}\binom{m}{r}(k+2)\cdots(k+3-r)(\lambda^*)^{k+2-r}(D_{\hat{x}}\lambda^*)^r D_{\hat{x}}^{m-r}(p(y)-y), \end{aligned}$$

so that applying (4.6),

$$\|D_{\hat{x}}^m((\lambda^*)^{k+2}(p(y)-y))\|_{L^\infty(\hat{T}\setminus\hat{\sigma})} \leqslant c\sum_{r=0}^{m}(\lambda^*)^{k+2-r}\frac{ch^2}{(\lambda^*)^{m-r}} \leqslant ch^2(\lambda^*)^{k+2-m}.$$

The mapping Φ_T is of class C^{k+1} on $\hat{T}\setminus\hat{\sigma}$ with derivatives of order less than or equal to $k+1$ tending to zero when $\hat{x}$ tends to a point in $\hat{\sigma}$. Hence, it can be extended to a C^{k+1} mapping on $\hat{T}$ (Gilbarg & Trudinger, 1983) which satisfies (4.7).

Since $|\partial\hat{x}_l/\partial x_j| \leqslant c/h$ (Ciarlet & Raviart, 1972a, p. 239), we know that

$$|A_T^{-1}| = \frac{c}{h}.$$

This result together with (4.7) shows

$$C_T \leqslant \sup_{\hat{x}\in\hat{T}} |D\Phi_T(\hat{x})||A_T^{-1}| \leqslant ch,$$

hence Φ_T satisfies (4.3) for h small enough. □

REMARK 4.5 Note that we could have chosen $\Phi_T(\hat{x}) = \lambda^*(p(y)-y)$. However, this function is not $C^1(T)$, and the interpolation theory of Bernardi (1989) would be unavailable. Our construction is a combination of ideas from Lenoir (1986) and Dubois (1990).

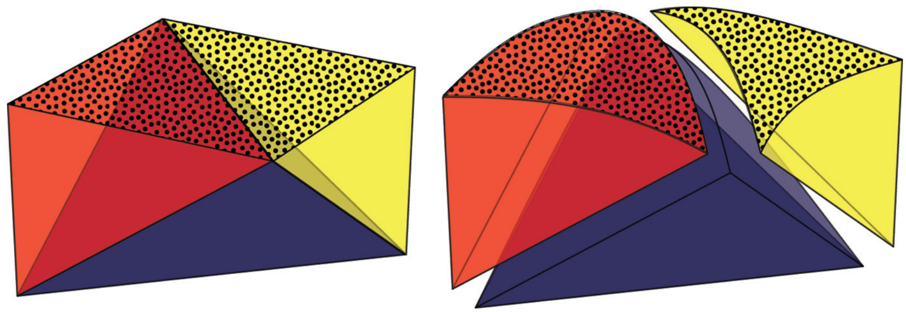

FIG. 2. A plot of two sections of triangulations. The left shows three tetrahedra in $\check{\mathscr{T}}_h$ and the right shows the corresponding three tetrahedra in $\mathscr{T}_h^{\mathrm{e}}$. The surface is shown by spots on both sides. The red and yellow tetrahedra (left and right in each image) share a face with the boundary ($l = 3$) and the blue tetrahedron (centre in each image) shares an edge with the boundary ($l = 2$). This means that the red and yellow curved tetrahedra have four curved faces and the blue tetrahedron has two curved faces.

We will call the exact triangulation, defined by F_T^{e} above, $\mathscr{T}_h^{\mathrm{e}}$. Note that under this construction, simplices in $\mathscr{T}_h^{\mathrm{e}}$, which have more than one vertex on the boundary, can have more than one curved face. See Fig. 2, for example.

4.1.2 *Computational domain.* We can now define our computational domains $\Omega_h^{(k)}$ and $\Gamma_h^{(k)}$. Let $T \in \check{\mathscr{T}}_h$ and $\phi_1^k, \dots, \phi_{n_k}^k$ be a Lagrangian basis of degree k on $\hat{T}$ corresponding to the nodal points $\hat{x}^1, \dots, \hat{x}^{n_k}$. Then for $\hat{x} \in \hat{T}$, we can define a parametrization of a polynomial simplex $T^{(k)}$ by

$$F_T^{(k)}(\hat{x}) = \sum_{j=1}^{n_k} F_T^{\mathrm{e}}(\hat{x}^j)\phi_j^k(\hat{x}).$$

We can carry out this procedure for each simplex $T \in \check{\mathscr{T}}_h$. Since the basis functions $\{\phi_j^k\}$ are unisolvent, $F_T^{(k)}$ is also a diffeomorphism. We define $\Omega_h^{(k)}$ as the union of elements $\mathscr{T}_h^{(k)}$ given by

$$T^{(k)} := \{F_T^{(k)}(\hat{x}) : \hat{x} \in \hat{T}\}, \quad \mathscr{T}_h^{(k)} := \{T^{(k)} | T \in \check{\mathscr{T}}_h\}. \tag{4.8}$$

Then $\Gamma_h^{(k)}$ is the boundary of the domain $\Omega_h^{(k)}$ with the triangulation $\mathscr{T}_h^{(k)}|_{\Gamma_h^{(k)}}$. This construction admits quasi-uniform triangulations $\mathscr{T}_h^{(k)}$ and $\mathscr{T}_h^{(k)}|_{\Gamma_h^{(k)}}$ for $\Omega_h^{(k)}$ and $\Gamma_h^{(k)}$, respectively. Note that, like the exact simplices in $\mathscr{T}_h^{\mathrm{e}}$, the simplices in $\mathscr{T}_h^{(k)}$ can have curved (polynomial) faces.

4.2 *Bulk estimates*

We define a function $G_h \colon \Omega_h^{(k)} \to \Omega$ locally by $G_h|_{T^{(k)}} := F_T^{\mathrm{e}} \circ (F_T^{(k)})^{-1}$ for each $T^{(k)} \in \mathscr{T}_h^{(k)}$. This is a homeomorphism, which when restricted to interior simplices (those with at most one vertex on the boundary) is the identity.

We use the notation DG_h for the gradient of G_h, where $(DG_h)_{ij} = (\partial/\partial x_j)(G_h)_i$, and DG_h^{T} for its transpose. We will also write DG_h^{-1} for $D(G_h^{-1}) = (DG_h)^{-1}$. We denote by $J_h|_T$ the absolute value of the determinant of $DG_h|T$.

We denote by B_h the union of elements in $\mathscr{T}_h^{(k)}$ which have more than one vertex on the boundary $\Gamma_h^{(k)}$ and B_h^ℓ the associated exact elements in $\mathscr{T}_h^{\mathrm{e}}$. Note that B_h is the region where G_h is different from the identity.

Let us use the notation that for a fixed $\hat{x} \in \hat{T}$, we denote $F_T^{(k)}(\hat{x}) = x$; then one may write that

$$G_h(x) = F_T^{\mathrm{e}}((F_T^{(k)})^{-1}(x)) = F_T^{\mathrm{e}}(\hat{x}) = x + (F_T^{\mathrm{e}}(\hat{x}) - F_T^{(k)}(\hat{x})). \tag{4.9}$$

Lemma 4.6 If Γ is C^{k+1}, then $G_h|_T \in C^{k+1}(T^{(k)})$ for each $T^{(k)} \in \mathscr{T}_h^{(k)}$ and we have that $\|G_h\|_{W^{k+1,\infty}(T^{(k)})}$ is bounded independently of h.

Proof. Using (4.9), we can write G_h as

$$G_h(x) = F_T(\hat{x}) + \Phi_T(\hat{x}).$$

Since $x \mapsto \hat{x}$ is smooth, G_h is the sum of an affine function and a C^{k+1} function, so G_h is of class C^{k+1} on $T^{(k)}$. To achieve the bound independently of h, we use (4.3). □

Proposition 4.7 (Geometric bulk estimates) Let $T \in \mathscr{T}_h^{(k)}$ be a boundary simplex (one which has more than one vertex on the boundary $\Gamma_h^{(k)}$), and T^{e} the associated exact triangle in $\mathscr{T}_h^{\mathrm{e}}$. Under the assumption that $\mathscr{T}_h$ is quasi-uniform, for sufficiently small h, we have that

$$\|DG_h^{\mathrm{T}}|_T - \mathrm{Id}\|_{L^\infty(T)} \leqslant ch^k, \tag{4.10a}$$

$$\|J_h|_T - 1\|_{L^\infty(T)} \leqslant ch^k. \tag{4.10b}$$

Proof. We will bound

$$\left|\frac{\partial}{\partial x_j}(G_h)_i - \delta_{ij}\right|,$$

which will show the estimates above.

We start by taking the x_j derivative of G_h to get

$$\frac{\partial}{\partial x_j}(G_h)_i = \sum_l \frac{\partial (F_T^{(k)})^{-1}(x)_l}{\partial x_j}\frac{\partial (F_T^{\mathrm{e}}(\hat{x}))_i}{\partial \hat{x}_l},$$

where we have used the substitution $F_T^{(k)}(\hat{x}) = x$. We note that this means

$$\sum_l \frac{\partial (F_T^{(k)})^{-1}(x)_l}{\partial x_j}\frac{\partial (F_T^{(k)}(\hat{x}))_i}{\partial \hat{x}_l} = \frac{\partial (F_T^{(k)})^{-1}(x)}{\partial x_j} = \delta_{ij}.$$

Hence

$$\frac{\partial}{\partial x_j}(G_h)_i - \delta_{ij} = \sum_l \frac{\partial (F_T^{(k)})^{-1}(x)_l}{\partial x_j}\frac{\partial}{\partial \hat{x}_l}(F_T^{\mathrm{e}}(\hat{x}) - F_T^{(k)}(\hat{x}))_i.$$

It is classical (Ciarlet & Raviart, 1972a, Lemma 7, p. 238) that

$$\left|\frac{\partial ((F_T^{(k)})^{-1}(\hat{x}))_l}{\partial x_j}\right| = \left|\frac{\partial \hat{x}_l}{\partial x_j}\right| \leqslant \frac{c}{h},$$

and from standard interpolation theory, we see that

$$\left| \frac{\partial}{\partial \hat{x}_l} (F_T^{\mathrm{e}}(\hat{x}) - F_T^{(k)}(\hat{x}))_i \right| \leqslant c \| D_{\hat{x}}^{k+1} (F_T^{\mathrm{e}}) \|_{L^\infty(\hat{T})}.$$

However, we may use the fact that $|D_{\hat{x}}^{m+1} x_j| \leqslant ch^m$ (Ciarlet & Raviart, 1972a, p. 239) and change coordinates to see

$$\| D_{\hat{x}}^{k+1} (F_T^{\mathrm{e}}) \|_{L^\infty(\hat{T})} \leqslant ch^{k+1} \| (F_T^{\mathrm{e}} \circ (F_T^{(k)})^{-1}) \|_{W^{k+1,\infty}(T^{(k)})} = ch^{k+1} \| G_h \|_{W^{k+1,\infty}(T^{(k)})}.$$

From Lemma 4.6, we know $\|G_h\|_{W^{k+1,\infty}(T^{(k)})}$ is bounded independently of h, this shows that

$$\left| \frac{\partial}{\partial x_j} (G_h)_i - \delta_{ij} \right| \leqslant ch^k.$$

□

We can now lift a function defined on $\Omega_h^{(k)}$ onto a function defined on Ω.

DEFINITION 4.8 For a function $\eta_h \colon \Omega_h^{(k)} \to \mathbb{R}$, we define its lift $\eta_h^\ell : \Omega \to \mathbb{R}$ by

$$\eta_h^\ell := \eta_h \circ G_h^{-1}.$$

For a function $\eta : \Omega \to \mathbb{R}$, we can also define an inverse lift $\eta^{-\ell} : \Omega_h^{(k)} \to \mathbb{R}$ by

$$\eta^{-\ell} := \eta \circ G_h.$$

In this case, it follows that $(\eta^{-\ell})^\ell = \eta$.

We also have equivalence of norms via this lifting process.

PROPOSITION 4.9 Let $\eta_h \colon \Omega_h^{(k)} \to \mathbb{R}$ and let $\eta_h^\ell \colon \Omega \to \mathbb{R}$ be its lift. Then there exist constants c_1, c_2, independent of h, such that

$$c_1 \| \eta_h^\ell \|_{L^2(\Omega)} \leqslant \| \eta_h \|_{L^2(\Omega_h^{(k)})} \leqslant c_2 \| \eta_h^\ell \|_{L^2(\Omega)}, \tag{4.11a}$$

$$c_1 \| \nabla \eta_h^\ell \|_{L^2(\Omega)} \leqslant \| \nabla \eta_h \|_{L^2(\Omega_h^{(k)})} \leqslant c_2 \| \nabla \eta_h^\ell \|_{L^2(\Omega)}. \tag{4.11b}$$

Proof. We can write integrals over $\Omega_h^{(k)}$ in the following way:

$$\int_{\Omega_h^{(k)}} \eta_h(x) \, \mathrm{d}x = \int_\Omega \eta_h^\ell(y) \frac{1}{J_h^\ell(y)} \, \mathrm{d}y,$$

and the gradient on $\Omega_h^{(k)}$ as

$$\nabla_x \eta_h(x) = DG_h^{\mathrm{T}}(y) \nabla_y \eta_h^\ell(y).$$

The results follow simply from applying the previous proposition. □

In the subsequent error analysis, we will require the following narrow band trace inequality.

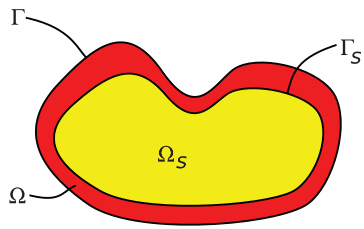

FIG. 3. A cartoon of the setup of Ω_s (yellow) and Γ_s lying inside Ω (red).

LEMMA 4.10 Let $\mathcal{N}_\delta \subseteq U$ be the band of width $\delta < \delta_\Gamma$ given by

$$\mathcal{N}_\delta = \{x \in \Omega : -\delta < d(x) < 0\}.$$

It holds that for $\eta \in H^1(\Omega)$

$$\|\eta\|_{L^2(\mathcal{N}_\delta)} \leqslant c\delta^{1/2}\|\eta\|_{H^1(\Omega)}. \tag{4.12}$$

Proof. First, we may assume that $\eta \in C^1(\Omega)$, since the more general result will follow by a density argument. Note that $d \in C^2(\mathcal{N}_\delta)$ and $|\nabla d| \equiv 1$ on $\mathcal{N}_\delta$. We can apply the co-area formula to integrals over $\mathcal{N}_\delta$ as follows:

$$\begin{aligned}\int_{\mathcal{N}_\delta} \eta(y)^2 \,\mathrm{d}y &= \int_{\mathcal{N}_\delta} \eta(y)^2 |\nabla d(y)| \,\mathrm{d}y \\ &= \int_{-\delta}^{0} \int_{\Gamma_s} \eta^2|_{\Gamma_s} \,\mathrm{d}o \,\mathrm{d}s.\end{aligned}$$

Here Γ_s denotes the C^2 hypersurface which is the inverse image of s under d, namely, $\Gamma_s = \{x \in \mathcal{N}_\delta : d(x) = s\}$. Next, we wish to apply a trace-inequality type argument to bound the right-hand side of this equation. We follow the proof of the trace inequality from Grisvard (2011, Theorem 1.5.1.10). Let the vector field $D\colon \bar{\Omega} \to \mathbb{R}^N$ be an extension of ∇d of class C^1 on $\bar{\Omega}$, equal to ∇d on $\mathcal{N}_\delta$, with the bound $\|D\|_{C^1(\bar{\Omega})} \leqslant c\|d\|_{C^2(\mathcal{N}_\delta)}$. Setting $\Omega_s = \{x \in \Omega : d(x) < s\}$, (see Fig. 3), we have that

$$\int_{\Omega_s} \nabla(\eta^2) \cdot D \,\mathrm{d}x = 2\int_{\Omega_s} \eta\nabla\eta \cdot D \,\mathrm{d}x.$$

On the other hand, applying Green's theorem, using the notation n_s for the normal to Γ_s, we obtain

$$\int_{\Omega_s} \nabla(\eta^2) \cdot D \,\mathrm{d}x = \int_{\Gamma_s} \eta^2 D \cdot \mathrm{n}_s \,\mathrm{d}o - \int_{\Omega_s} \eta^2 \nabla \cdot D \,\mathrm{d}x.$$

Since $D \cdot \mathrm{n}_s = 1$ on Γ_s, combining these two equations we have that

$$\int_{\Gamma_s} \eta^2 D \cdot \mathrm{n}_s \,\mathrm{d}o = 2\int_{\Omega_s} \eta\nabla\eta \cdot D \,\mathrm{d}x + \int_{\Omega_s} \eta^2\nabla \cdot D \,\mathrm{d}x,$$

which means that

$$\int_{\Gamma_s} \eta^2 \,\mathrm{d}o \leqslant 2 \max_{\bar{\Omega}_s} |D| \int_{\Omega_s} |\eta||\nabla \eta| \,\mathrm{d}x + \max_{\bar{\Omega}_s} |\nabla \cdot D| \int_{\Omega_s} \eta^2 \,\mathrm{d}x.$$

Since we have that $\Omega_s \subseteq \Omega$, applying Young's inequality gives

$$\int_{\Gamma_s} \eta^2 \,\mathrm{d}o \leqslant c\|D\|_{C^1(\bar{\Omega})} \int_{\Omega} |\nabla \eta^2| + \eta^2 \,\mathrm{d}x.$$

Hence we have that

$$\int_{\mathcal{N}_\delta} \eta^2 \,\mathrm{d}y \leqslant c\delta \|\eta\|^2_{H^1(\Omega)}. \tag{4.13}$$

□

4.3 *Surface estimates*

We have the following geometric estimates for the surface Γ_h. They follow since Γ_h can be viewed as an interpolant of Γ. Details can be found in Dziuk (1988), Dziuk & Elliott (2007a), Dziuk & Elliott (2007b) and Demlow (2009).

PROPOSITION 4.11 (Geometric surface estimates) Under the above assumptions on Γ and Γ_h, we have that

$$\|d\|_{L^\infty(\Gamma_h^{(k)})} \leqslant ch^{k+1}.$$

Let μ_h be the quotient of the measures on the surface and the approximate surface, so that $\mathrm{d}o = \mu_h \,\mathrm{d}o_h$. Then we have the estimate

$$\sup_{\Gamma_h^{(k)}} |1 - \mu_h| \leqslant ch^{k+1}. \tag{4.14}$$

Let P and P_h denote the projections onto the tangent spaces of Γ and Γ_h, respectively. We introduce the notation

$$\mathcal{Q}_h = \frac{1}{\mu_h}(\mathrm{Id} - d\mathcal{H})PP_hP(\mathrm{Id} - d\mathcal{H}), \tag{4.15}$$

then we have the estimate that

$$|\mathrm{Id} - \mu_h \mathcal{Q}_h| \leqslant ch^{k+1}. \tag{4.16}$$

A proof can be found in Dziuk (1988) and Dziuk & Elliott (2007a) for the linear case and Demlow (2009) for higher orders.

We use the closest point operator (2.1) to define the lift and inverse lift of surface functions.

DEFINITION 4.12 Given $\xi_h \colon \Gamma_h^{(k)} \to \mathbb{R}$, we define its lift, denoted by $\xi_h^\ell \colon \Gamma \to \mathbb{R}$, by

$$\xi_h^\ell(p(x)) := \xi_h(x).$$

Similarly, for a function $\xi \colon \Gamma \to \mathbb{R}$, we define its inverse lift, written $\xi^{-\ell} \colon \Gamma_h^{(k)} \to \mathbb{R}$, by

$$\xi^{-\ell}(x) := \xi(p(x)).$$

It can be shown that the following norms are equivalent via this lifting process (see Fig. 4).

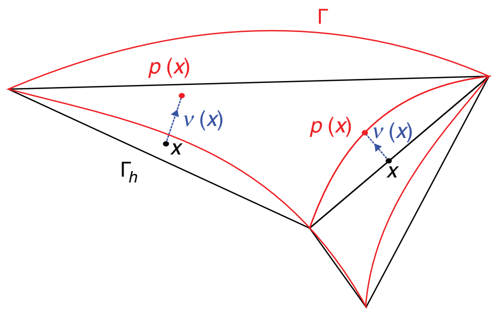

FIG. 4. A section of a surface triangulation with normal lifts shown in $\mathbb{R}^3$.

PROPOSITION 4.13 Let $\xi_h\colon \Gamma_h^{(k)} \to \mathbb{R}$ and let $\xi_h^\ell\colon \Gamma \to \mathbb{R}$ be its lift. Then there exist constants c_1, c_2, independent of h, such that

$$c_1 \|\xi_h^\ell\|_{L^2(\Gamma)} \leqslant \|\xi_h\|_{L^2(\Gamma_h^{(k)})} \leqslant c_2 \|\xi_h^\ell\|_{L^2(\Gamma)}, \tag{4.17a}$$

$$c_1 \|\nabla_\Gamma \xi_h^\ell\|_{L^2(\Gamma)} \leqslant \|\nabla_{\Gamma_h} \xi_h\|_{L^2(\Gamma_h^{(k)})} \leqslant c_2 \|\nabla_\Gamma \xi_h^\ell\|_{L^2(\Gamma)}. \tag{4.17b}$$

A proof is given in Dziuk (1988), Dziuk & Elliott (2007a) for $k = 1$ and Demlow (2009) for any $k > 1$.

5. Finite element method

In this work we will use piecewise polynomial finite element functions of the same degree as the approximation of the domain. This leads to so-called isoparametric elements which will give the optimal rate of convergence. One could also implement this method with different order finite element functions, but this would lead to suboptimal convergence.

5.1 *Isoparametric finite element spaces*

We use this section to define the finite element spaces V_h and S_h that our finite element method will be based on. We recall that the computational domains Ω_h and Γ_h are defined elementwise by a parametrization $F_T^{(k)}\colon \hat{T} \to T^{(k)} \subset \Omega_h^{(k)}$ as in (4.8). In both the bulk and surface cases, we define the finite element functions to be continuous functions which are piecewise polynomials of degree k with respect to the barycentric coordinates of the reference element in dimensions N and $N-1$. An important part of the construction is that the trace of a function on $\Gamma_h^{(k)}$ in V_h lies in S_h.

More precisely, for the bulk finite element functions,

$$V_h = \{\eta_h \in C(\Omega_h^{(k)}) : \eta_h|_T = \hat{\eta}_h \circ (F_T^{(k)})^{-1} \text{ with } \hat{\eta}_h \in P_k(\hat{T}) \text{ for all } T \in \mathscr{T}_h\}.$$

For the surface finite element functions, we introduce

$$S_h = \{\xi_h \in C(\Gamma_h^{(k)}) : \xi_h|_\tau = \hat{\xi}_h \circ (F_T^{(k)})^{-1} \text{ with } \hat{\xi}_h \in P_k(\hat{\tau}) \text{ for all } T \in \mathscr{T}_h \text{ with } \tau = T \cap \Gamma_h \neq \emptyset\}.$$

We have used the notation $\hat{\tau} = (F_T^{(k)})^{-1}(\tau)$ for the face of the reference element $\hat{T}$ corresponding to τ, and $P_k(\omega)$ for the space of polynomials of degree k on ω.

From now on we will assume k is fixed and write Ω_h, Γ_h, $\mathscr{T}_h$ for $\Omega_h^{(k)}$, $\Gamma_h^{(k)}$, $\mathscr{T}_h^{(k)}$, without ambiguity.

5.2 *Description of the method*

We define approximate data f_h, g_h using the appropriate inverse lifts. That is,

$$f_h = f^{-\ell} J_h, \quad g_h = g^{-\ell} \mu_h. \tag{5.1}$$

The approximate problem is then to find $(u_h, v_h) \in V_h \times S_h$ such that

$$\begin{aligned} &\alpha \int_{\Omega_h} \nabla u_h \cdot \nabla \eta_h + u_h \eta_h \,\mathrm{d}x + \beta \int_{\Gamma_h} \nabla_{\Gamma_h} v_h \cdot \nabla_{\Gamma_h} \xi_h + v_h \xi_h \,\mathrm{d}o_h \\ &\quad + \int_{\Gamma_h} (\alpha u_h - \beta v_h)(\alpha \eta_h - \beta \xi_h) \,\mathrm{d}o_h = \alpha \int_{\Omega_h} f_h \eta_h \,\mathrm{d}x + \beta \int_{\Gamma_h} g_h \xi_h \,\mathrm{d}o_h \\ &\quad \text{for all } (\eta_h, \xi_h) \in V_h \times S_h, \end{aligned} \tag{5.2}$$

where ∇_{Γ_h} is the surface gradient on Γ_h.

REMARK 5.1 This choice of f_h and g_h is not fully practical for arbitrary $(f, g) \in L^2(\Omega) \times L^2(\Gamma)$ as the right-hand side integrals would need to be calculated via some numerical integration rule. We are not concerned in analysing such errors in this paper and will assume that it is possible to calculate these integrals exactly. For general results on numerical integration in the context of curved domains, see Ciarlet & Raviart (1972b) and Barrett & Elliott (1987).

REMARK 5.2 To implement the method, we use exact quadrature rules to calculate mass and stiffness matrices on reference elements using the transformation (4.8).

We introduce bilinear and linear forms on $V_h \times S_h$:

$$\begin{aligned} a_h((w_h, y_h), (\eta_h, \xi_h)) &= \alpha \int_{\Omega_h} \nabla w_h \cdot \nabla \eta_h + w_h \eta_h \,\mathrm{d}x \\ &\quad + \beta \int_{\Gamma_h} \nabla_{\Gamma_h} y_h \cdot \nabla_{\Gamma_h} \xi_h + y_h \xi_h \,\mathrm{d}o_h \\ &\quad + \int_{\Gamma_h} (\alpha w_h - \beta y_h)(\alpha \eta_h - \beta \xi_h) \,\mathrm{d}o_h, \\ l_h((\eta_h, \xi_h)) &= \alpha \int_{\Omega_h} f_h \eta_h \,\mathrm{d}x + \beta \int_{\Gamma_h} g_h \xi_h \,\mathrm{d}o_h, \end{aligned}$$

so that we can write (5.2) as: find $(u_h, v_h) \in V_h \times S_h$ such that

$$a_h((u_h, v_h), (\eta_h, \xi_h)) = l_h((\eta_h, \xi_h)) \quad \text{for all } (\eta_h, \xi_h) \in V_h \times S_h. \tag{5.3}$$

THEOREM 5.3 The finite element method defined in (5.2) has a unique solution $(u_h, v_h) \in V_h \times S_h$ which satisfies the bound

$$\|(u_h, v_h)\|_{H^1(\Omega_h)\times H^1(\Gamma_h)} \leqslant c\|(f, g)\|_{L^2(\Omega)\times L^2(\Gamma)}, \text{ for all } h. \tag{5.4}$$

Proof. It is clear that the equations have a unique solution since a_h is also coercive; This follows from the same reasoning as (3.2). To show the bound, we use the coercivity of a_h, the equivalence of norms shown in (4.17a), (4.11a), (4.14) and (4.10) to see that for h small enough,

$$\begin{aligned}\|(u_h, v_h)\|_{H^1(\Omega_h)\times H^1(\Gamma_h)} &\leqslant c\|(f_h, g_h)\|_{L^2(\Omega_h)\times L^2(\Gamma_h)}\\ &\leqslant c\|(f, g)\|_{L^2(\Omega)\times L^2(\Gamma)}.\end{aligned}$$

□

5.3 *Lifted finite element spaces*

In order to prove error bounds, we define the lifted finite element spaces that lifts of finite element functions live in. In particular, this allows us to define (u_h^ℓ, v_h^ℓ): the lifts of the finite element solution defined on the same domain as the solutions of the continuous problem. We define the lift of the finite element spaces as

$$\begin{aligned} V_h^\ell &= \{\eta_h^\ell : \eta_h \in V_h\} \subseteq H^1(\Omega),\\ S_h^\ell &= \{\xi_h^\ell : \xi_h \in S_h\} \subseteq H^1(\Gamma). \end{aligned} \tag{5.5}$$

It is important to note that the traces on Γ of functions in V_h^ℓ live in S_h^ℓ.

PROPOSITION 5.4 (Approximation property) For the lifted finite element spaces V_h^ℓ, S_h^ℓ defined above, there exists an interpolation operator $I_h : H^{k+1}(\Omega) \times H^{k+1}(\Gamma) \to V_h^\ell \times S_h^\ell$ such that for $2 \leqslant m \leqslant k+1$,

$$\|(w, y) - I_h(w, y)\|_{L^2(\Omega)\times L^2(\Gamma)} + h\|(w, y) - I_h(w, y)\|_{H^1(\Omega)\times H^1(\Gamma)} \leqslant ch^m\|(w, y)\|_{H^m(\Omega)\times H^m(\Gamma)} \tag{5.6}$$

for all $(w, y) \in H^2(\Omega) \times H^2(\Gamma)$.

Proof. We start by defining the interpolation operator $\widetilde{I}_h : H^2(\Omega) \times H^2(\Gamma) \to V_h \times S_h$ so that (w, y) and $\widetilde{I}_h(w, y)$ agree at the nodes of Ω_h and Γ_h. We use both lifts to define $I_h(w, y) = (\widetilde{I}_h(w, y))^\ell$. The error bounds follow from given interpolation theory; see Bernardi (1989, Corollary 4.1) for the bulk and Demlow (2009) for the surface. □

Using the fact that

$$\nabla(w_h^\ell) = \nabla(w_h \circ G_h^{-1}) = DG_h^{-\mathrm{T}}(\nabla w_h)^\ell,$$

(writing $DG_h^{-\mathrm{T}}$ for $(DG_h^{-1})^{\mathrm{T}}$) and from Dziuk (1988),

$$(P_h(\mathrm{Id} - d\mathcal{H}))\nabla_\Gamma(y_h^\ell) = (\nabla_{\Gamma_h} y_h)^\ell,$$

we have that

$$
\begin{aligned}
a_h((w_h, y_h), (\eta_h, \xi_h)) = \alpha & \int_\Omega (DG_h^{\mathrm{T}} \nabla w_h^\ell \cdot DG_h^{\mathrm{T}} \nabla \eta_h^\ell + w_h^\ell \eta_h^\ell) \frac{1}{J_h^\ell} \,\mathrm{d}x \\
& + \beta \int_\Gamma \mathcal{Q}_h^\ell \nabla_\Gamma y_h^\ell \cdot \nabla_\Gamma \xi_h^\ell + y_h^\ell \xi_h^\ell \frac{1}{\mu_h^\ell} \,\mathrm{d}o \\
& + \int_\Gamma (\alpha w_h^\ell - \beta y_h^\ell)(\alpha \eta_h^\ell - \beta \xi_h^\ell) \frac{1}{\mu_h^\ell} \,\mathrm{d}o \\
=: & \, a_h^\ell((w_h^\ell, y_h^\ell), (\eta_h^\ell, \xi_h^\ell)),
\end{aligned}
$$

for all $(w_h, y_h), (\eta_h, \xi_h) \in V_h \times S_h$ with lifts $(w_h^\ell, y_h^\ell), (\eta_h^\ell, \xi_h^\ell) \in V_h^\ell \times S_h^\ell$.

For the right-hand side, we immediately have that $l_h((\eta_h, \xi_h)) = l((\eta_h^\ell, \xi_h^\ell))$ since

$$
\int_{\Omega_h} f_h \eta_h \,\mathrm{d}x = \int_{\Omega_h} (f^{-\ell} J_h) \eta_h \,\mathrm{d}x = \int_\Omega (f^{-\ell} J_h)^\ell \eta_h^\ell \frac{1}{J_h^\ell} \,\mathrm{d}x = \int_\Omega f J_h^\ell \eta_h^\ell \frac{1}{J_h^\ell} \,\mathrm{d}x = \int_\Omega f \eta_h^\ell \,\mathrm{d}x,
$$

and

$$
\int_{\Gamma_h} g_h \xi_h \,\mathrm{d}o_h = \int_{\Gamma_h} (g^{-\ell} \mu_h) \xi_h \,\mathrm{d}o_h = \int_\Gamma (g^{-\ell} \mu_h)^\ell \xi_h^\ell \frac{1}{\mu_h^\ell} \,\mathrm{d}o = \int_\Gamma g \mu_h^\ell \xi_h^\ell \frac{1}{\mu_h^\ell} \,\mathrm{d}o = \int_\Gamma g \xi_h^\ell \,\mathrm{d}o.
$$

Hence, we may rewrite (5.3) as: find $(u_h^\ell, v_h^\ell) \in V_h^\ell \times S_h^\ell$ such that

$$
a_h^\ell((u_h^\ell, v_h^\ell), (\eta_h^\ell, \xi_h^\ell)) = l((\eta_h^\ell, \xi_h^\ell)) \quad \text{for all } (\eta_h^\ell, \xi_h^\ell) \in V_h^\ell \times S_h^\ell. \tag{5.7}
$$

In the following, we will make use of the fact that a_h^ℓ now makes sense for all function pairs in $H^1(\Omega) \times H^1(\Gamma)$

6. Error analysis

In this section, we wish to compare the error of the solutions (u, v) of the exact problem (1.1) to the solutions (u_h, v_h) of the approximate problem (5.2) defined in Section 5.

One of the problems we have to overcome is the fact that the two problems are posed over different domains. However, the lift operators we have defined will help us.

In order to derive optimal order estimates for $k > 1$, we must assume higher regularity of the smooth solution (u, v) of (2.10) and the surface Γ. We require $(u, v) \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)$ which requires Γ to be C^{k+2} (Wloka, 1987).

THEOREM 6.1 Let $(u, v) \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)$ be the solution of the variational problem (2.10) and let $(u_h, v_h) \in V_h \times S_h$ be the solution of the finite element scheme given by (5.2). Denote by u_h^ℓ and v_h^ℓ the lifts of u_h and v_h, respectively. Then we have the following error bounds:

$$
\|(u - u_h^\ell, v - v_h^\ell)\|_{H^1(\Omega) \times H^1(\Gamma)} \leqslant C_1 h^k, \tag{6.1}
$$

where

$$
C_1 = c(\|(u, v)\|_{H^{k+1}(\Omega) \times H^{k+1}(\Gamma)} + \|(f, g)\|_{L^2(\Omega) \times L^2(\Gamma)}),
$$

and

$$\|(u-u_h^\ell, v-v_h^\ell)\|_{L^2(\Omega)\times L^2(\Gamma)} \leqslant C_2 h^{k+1}, \tag{6.2}$$

where

$$C_2 = c(\|(u,v)\|_{H^{k+1}(\Omega)\times H^{k+1}(\Gamma)} + \|(f,g)\|_{L^2(\Omega)\times L^2(\Gamma)}).$$

6.1 *Geometric errors*

Part of the error of the finite element method comes from the fact that there is a so-called 'variational crime', that is, we are using different bilinear forms in the exact and approximate formulations and $V_h \not\subseteq H^1(\Omega)$ and $S_h \not\subseteq H^1(\Gamma)$. These errors come from the change in geometry of the computational domain.

LEMMA 6.2 For $(w,y),(\eta,\xi) \in V_h^\ell \times S_h^\ell$, we have

$$\begin{aligned} &|a((w,y),(\eta,\xi)) - a_h^\ell((w,y),(\eta,\xi))| \\ &\quad \leqslant ch^k \|w\|_{H^1(B_h^\ell)} \|\eta\|_{H^1(B_h^\ell)} + ch^{k+1} \|(w,y)\|_{H^1(\Omega)\times H^1(\Gamma)} \|(\eta,\xi)\|_{H^1(\Omega)\times H^1(\Gamma)}. \end{aligned} \tag{6.3}$$

Proof. To prove this lemma, we will split the forms a and a_h^ℓ into bulk, surface and cross terms. That is,

$$\begin{aligned} a^{(\Omega)}(w,\eta) &= \alpha \int_\Omega \nabla w \cdot \nabla \eta + w\eta \,\mathrm{d}x, \\ a^{(\Gamma)}(y,\xi) &= \beta \int_\Gamma \nabla_\Gamma y \cdot \nabla_\Gamma \xi + y\xi \,\mathrm{d}o, \\ a^{(\times)}((w,y),(\eta,\xi)) &= \int_\Gamma (\alpha w - \beta y)(\alpha\eta - \beta\xi)\,\mathrm{d}o. \end{aligned}$$

We define $a_h^{(\cdot)\ell}$ similarly.

Given $w, \eta \in V_h^\ell$, for the bulk term we see that

$$\left| \int_{\Omega_h} \nabla w^{-\ell} \cdot \nabla \eta^\ell \,\mathrm{d}x - \int_\Omega \nabla w \cdot \nabla \eta \,\mathrm{d}x \right| = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3,$$

where

$$\begin{aligned} \mathcal{A}_1 &= \int_\Omega (DG_h^{\mathrm{T}} - \mathrm{Id})\nabla w \cdot DG_h^{\mathrm{T}} \nabla \eta \frac{1}{J_h^\ell} \,\mathrm{d}x, \\ \mathcal{A}_2 &= \int_\Omega \nabla w \cdot (DG_h^{\mathrm{T}} - \mathrm{Id})\nabla \eta \frac{1}{J_h^\ell} \,\mathrm{d}x, \\ \mathcal{A}_3 &= \int_\Omega \nabla w \cdot \nabla \eta \left(\frac{1}{J_h^\ell} - 1 \right) \mathrm{d}x. \end{aligned}$$

Making use of the fact that

$$\frac{1}{J_h^\ell} - 1 = 0 \quad \text{and} \quad DG_h^{\mathrm{T}} - \mathrm{Id} = 0, \quad \text{in } \Omega \backslash B_h^\ell,$$

we actually have

$$\mathcal{A}_1 = \int_{B_h^\ell} (DG_h^{\mathrm{T}} - \mathrm{Id})\nabla w \cdot DG_h^{\mathrm{T}} \nabla \eta \frac{1}{J_h^\ell} \, \mathrm{d}x,$$
$$\mathcal{A}_2 = \int_{B_h^\ell} \nabla w \cdot (DG_h^{\mathrm{T}} - \mathrm{Id}) \nabla \eta \frac{1}{J_h^\ell} \, \mathrm{d}x,$$
$$\mathcal{A}_3 = \int_{B_h^\ell} \nabla w \cdot \nabla \eta \left(\frac{1}{J_h^\ell} - 1 \right) \mathrm{d}x.$$

Using Proposition 4.7, we see that the three terms $\mathcal{A}_j$ are bounded by

$$ch^k \|\nabla w\|_{L^2(B_h^\ell)} \|\nabla \eta\|_{L^2(\Omega)}.$$

Similarly,

$$\left| \int_{\Omega_h} w^{-\ell} \eta^{-\ell} \, \mathrm{d}x - \int_\Omega w\eta \, \mathrm{d}x \right| = \left| \int_\Omega w\eta \left(\frac{1}{J_h^\ell} - 1 \right) \mathrm{d}x \right) \right| \leqslant ch^k \|w\|_{L^2(\Omega)} \|\eta\|_{L^2(\Omega)}.$$

Given $y, \xi \in S_h^\ell$, using Proposition 4.11, we see that for surface terms,

$$\left| \int_{\Gamma_h} \nabla_{\Gamma_h} y^{-\ell} \cdot \nabla_{\Gamma_h} \xi^{-\ell} \, \mathrm{d}o_h - \int_\Gamma \nabla_\Gamma y \cdot \nabla_\Gamma \xi \, \mathrm{d}o \right|$$
$$= \left| \int_\Gamma (\mathrm{Id} - \mu_h^\ell \mathcal{Q}_h^\ell) \nabla_\Gamma y \cdot \nabla_\Gamma \xi \, \mathrm{d}o \right| \leqslant ch^{k+1} \|\nabla_\Gamma y\|_{L^2(\Gamma)} \|\nabla_\Gamma \xi\|_{L^2(\Gamma)},$$

and

$$\left| \int_{\Gamma_h} y^{-\ell} \xi^{-\ell} \, \mathrm{d}o_h - \int_\Gamma y\xi \, \mathrm{d}o \right| = \left| \int_\Gamma y\xi \left(\frac{1}{\mu_h^\ell} - 1 \right) \mathrm{d}o \right| \leqslant ch^{k+1} \|y\|_{L^2(\Gamma)} \|\xi\|_{L^2(\Gamma)}.$$

Using the previous result, we also have that

$$\left| \int_{\Gamma_h} (\alpha w^{-\ell} - \beta y^{-\ell})(\alpha \eta^{-\ell} - \beta \xi^{-\ell}) \, \mathrm{d}o_h - \int_\Gamma (\alpha w - \beta y)(\alpha \eta - \beta \xi) \, \mathrm{d}o \right|$$
$$= \left| \int_\Gamma (\alpha w - \beta y)(\alpha \eta - \beta \xi) \left(\frac{1}{\mu_h^\ell} - 1 \right) \mathrm{d}o \right|$$
$$\leqslant ch^{k+1} \|(w, y)\|_{L^2(\Gamma) \times L^2(\Gamma)} \|(\eta, \xi)\|_{L^2(\Gamma) \times L^2(\Gamma)}$$
$$\leqslant ch^{k+1} \|(w, y)\|_{H^1(\Omega) \times H^1(\Gamma)} \|(\eta, \xi)\|_{H^1(\Omega) \times H^1(\Gamma)}.$$

This shows (6.3). □

We remark briefly that since B_h^ℓ is contained in Ω, we also have for functions $(\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma)$,

$$|a((w, y), (\eta, \xi)) - a_h^\ell((w, y), (\eta, \xi))|$$
$$\leqslant ch^k \|(w, y)\|_{H^1(\Omega) \times H^1(\Gamma)} \|(\eta, \xi)\|_{H^1(\Omega) \times H^1(\Gamma)}. \tag{6.4}$$

Finally, we remark that we can use Lemma 4.10 for integrals over B_h^ℓ.

LEMMA 6.3 For $\eta \in H^1(\Omega)$,

$$\|\eta\|_{L^2(B_h^\ell)} \leqslant ch^{1/2}\|\eta\|_{H^1(\Omega)}. \tag{6.5}$$

Proof. We may apply Lemma 4.10 to the domain $\mathcal{N}_\delta$. We can choose δ such that $\delta_\Gamma > ch > \delta > h > 0$, since the width of B_h^ℓ is just one element. Hence

$$\|\eta\|_{L^2(B_h^\ell)} \leqslant \|\eta\|_{L^2(\mathcal{N}_\delta)} \leqslant c\delta^{1/2}\|\eta\|_{H^1(\Omega)} \leqslant ch^{1/2}\|\eta\|_{H^1(\Omega)}. \qquad \square$$

6.2 *Proof of error bounds*

Let $(u, v) \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)$ be the solution of the variational problem (2.6) and let $(u_h, v_h) \in V_h \times S_h$ be the solution of the finite element scheme given by (5.2). Denote by u_h^ℓ and v_h^ℓ the lifts of u_h and v_h, respectively. Define $F_h : H^1(\Omega) \times H^1(\Gamma) \to \mathbb{R}$ by

$$F_h((\eta, \xi)) := a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi)). \tag{6.6}$$

LEMMA 6.4 If $(\eta, \xi) = (\eta_h^\ell, \xi_h^\ell) \in V_h^\ell \times S_h^\ell$, then F_h is bounded by

$$|F_h((\eta_h^\ell, \xi_h^\ell))| \leqslant ch^k \|(u_h^\ell, v_h^\ell)\|_{H^1(\Omega)\times H^1(\Gamma)} \|(\eta_h^\ell, \xi_h^\ell)\|_{H^1(\Omega)\times H^1(\Gamma)}. \tag{6.7}$$

If $(\eta, \xi) \in H^2(\Omega) \times H^2(\Gamma)$, then we can improve the bound on F_h to

$$\begin{aligned} |F_h(\eta, \xi)| \leqslant (&ch^{k+1}\|(u_h^\ell, v_h^\ell)\|_{H^1(\Omega)\times H^1(\Gamma)} + ch^k\|(u_h^\ell - u, v_h^\ell - v)\|_{H^1(\Omega)\times H^1(\Gamma)} \\ &+ ch^{k+1}\|(u, v)\|_{H^2(\Omega)\times H^2(\Gamma)})\|(\eta, \xi)\|_{H^2(\Omega)\times H^2(\Gamma)}. \end{aligned} \tag{6.8}$$

Proof. First, we note that if $(\eta, \xi) = (\eta_h^\ell, \xi_h^\ell) \in V_h^\ell \times S_h^\ell$, using the fact that (u, v) satisfies (2.6) and (u_h^ℓ, v_h^ℓ) satisfies (5.7), F_h can be written as

$$\begin{aligned} F_h((\eta_h^\ell, \xi_h^\ell)) &= a((u - u_h^\ell, v - v_h^\ell), (\eta_h^\ell, \xi_h^\ell)) \\ &= l((\eta_h^\ell, \xi_h^\ell)) - a((u_h^\ell, v_h^\ell), (\eta_h^\ell, \xi_h^\ell)) \\ &= (l((\eta_h^\ell, \xi_h^\ell)) - l((\eta_h^\ell, \xi_h^\ell))) \\ &\quad - (a((u_h^\ell, v_h^\ell), (\eta_h^\ell, \xi_h^\ell)) - a_h^\ell((u_h^\ell, v_h^\ell), (\eta_h^\ell, \xi_h^\ell))) \\ &= -(a((u_h^\ell, v_h^\ell), (\eta_h^\ell, \xi_h^\ell)) - a_h^\ell((u_h^\ell, v_h^\ell), (\eta_h^\ell, \xi_h^\ell))). \end{aligned}$$

Applying the result from (6.4) gives (6.7).

To show the second result, we assume $(\eta, \xi) \in H^2(\Omega) \times H^2(\Gamma)$ and introduce the interpolant $I_h(\eta, \xi) \in V_h^\ell \times S_h^\ell$ of (η, ξ), so that

$$\begin{aligned} F_h((\eta, \xi)) &= a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi)) \\ &= a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi) - I_h(\eta, \xi)) + a((u - u_h^\ell, v - v_h^\ell), I_h(\eta, \xi)). \end{aligned}$$

Then, again we can use the fact that (u, v) satisfies (2.6) and (u_h^ℓ, v_h^ℓ) satisfies (5.7), so that

$$F_h((\eta, \xi)) = a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi) - I_h(\eta, \xi)) + (a_h^\ell((u_h^\ell, v_h^\ell), I_h(\eta, \xi)) - a((u_h^\ell, v_h^\ell), I_h(\eta, \xi))).$$

Hence we have that

$$\begin{aligned} F_h((\eta,\xi)) = {} & a((u-u_h^\ell, v-v_h^\ell),(\eta,\xi)-I_h(\eta,\xi)) \\ & + (a_h^\ell((u_h^\ell,v_h^\ell),I_h(\eta,\xi)-(\eta,\xi)) - a((u_h^\ell,v_h^\ell),I_h(\eta,\xi)-(\eta,\xi))) \\ & + (a_h^\ell((u_h^\ell-u,v_h^\ell-v),(\eta,\xi)) - a((u_h^\ell-u,v_h^\ell-v),(\eta,\xi))) \\ & + (a_h^\ell((u,v),(\eta,\xi)) - a((u,v),(\eta,\xi))). \end{aligned} \tag{6.9}$$

We bound each of the terms on the right-hand side of (6.9) in turn. For the first term we apply (6.1) together with the approximation property (Proposition 5.4) to see

$$|a((u-u_h^\ell, v-v_h^\ell),(\eta,\xi)-I_h(\eta,\xi))| \leqslant C_1 h^k ch \|(\eta,\xi)\|_{H^2(\Omega)\times H^2(\Gamma)}.$$

For the second term, we use the geometric bound (6.4), again with the approximation property (Proposition 5.4) to get

$$\begin{aligned} & |a_h^\ell((u_h^\ell,v_h^\ell),I_h(\eta,\xi)-(\eta,\xi)) - a((u_h^\ell,v_h^\ell),I_h(\eta,\xi)-(\eta,\xi))| \\ & \quad \leqslant ch^k \|(u_h^\ell,v_h^\ell)\|_{H^1(\Omega)\times H^1(\Gamma)} ch \|(\eta,\xi)\|_{H^2(\Omega)\times H^2(\Gamma)}. \end{aligned}$$

A bound for the third term follows by applying the geometric bound (6.4):

$$\begin{aligned} & |a_h^\ell((u_h^\ell-u,v_h^\ell-v),(\eta,\xi)) - a((u_h^\ell-u,v_h^\ell-v),(\eta,\xi))| \\ & \quad \leqslant ch^k \|(u_h^\ell-u,v_h^\ell-v)\|_{H^1(\Omega)\times H^1(\Gamma)} \|(\eta,\xi)\|_{H^1(\Omega)\times H^1(\Gamma)}. \end{aligned}$$

Finally, for the fourth term, we simply apply (6.3) followed by the result from Lemma 6.3 to see

$$\begin{aligned} & |a_h^\ell((u,v),(\eta,\xi)) - a((u,v),(\eta,\xi))| \\ & \quad \leqslant ch^k \|u\|_{H^1(B_h^\ell)} \|\eta\|_{H^1(B_h^\ell)} + ch^{k+1} \|(u,v)\|_{H^1(\Omega)\times H^1(\Gamma)} \|(\eta,\xi)\|_{H^1(\Omega)\times H^1(\Gamma)} \\ & \quad \leqslant ch^{k+1} \|(u,v)\|_{H^2(\Omega)\times H^2(\Gamma)} \|(\eta,\xi)\|_{H^2(\Omega)\times H^2(\Gamma)}. \end{aligned}$$

Adding the previous four results into (6.9) gives (6.8). □

REMARK 6.5 Note that for $(\eta,\xi) = (\eta_h,\xi_h) \in V_h \times S_h$, in the absence of domain perturbation then

$$F_h((\eta_h,\xi_h)) = 0,$$

where this is simply Galerkin orthogonality, whereas in the absence of the bulk equations then the bound would be of order h^{k+1} (see Demlow 2009).

Proof of Theorem 6.1. The error estimate (6.1) follows simply by combining the approximation property (Proposition 5.4) with the bound on F_h from (6.7). We rewrite the error as

$$\begin{aligned} a((u - u_h^\ell, v - v_h^\ell), (u - u_h^\ell, v - v_h^\ell)) \\ = a((u - u_h^\ell, v - v_h^\ell), (u, v) - I_h(u, v)) \\ + a((u - u_h^\ell, v - v_h^\ell), I_h(u, v) - (u_h^\ell, v_h^\ell)) \\ = a((u - u_h^\ell, v - v_h^\ell), (u, v) - I_h(u, v)) + F_h(I_h(u, v) - (u_h^\ell, v_h^\ell)). \end{aligned}$$

The result follows from the application of a Cauchy inequality and the coercivity of the bilinear form a in (3.2). To show the given value of C_1 we use (5.4) from Theorem 5.3 and (4.17), (4.11) to bound $\|(u_h^\ell, v_h^\ell)\|_{H^1(\Omega)\times H^1(\Gamma)}$.

We will use an Aubin–Nitsche duality argument to show the L^2 bound. For $\zeta = (\zeta_1, \zeta_2) \in L^2(\Omega) \times L^2(\Gamma)$, we define the dual problem: find $z_\zeta \in H^1(\Omega) \times H^1(\Gamma)$ such that

$$a((\eta, \xi), z_\zeta) = \langle \zeta, (\eta, \xi)\rangle_{L^2(\Omega)\times L^2(\Gamma)} \quad \text{for all } (\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma). \tag{6.10}$$

Here, $\langle (w, y), (\eta, \xi)\rangle \in L^2(\Omega) \times L^2(\Gamma)$ denotes the sum of the L^2 inner products between w and η on Ω and y and ξ on Γ. Similarly to Theorem 3.2, one can show the following regularity result for the dual problem:

$$\|z_\zeta\|_{H^2(\Omega)\times H^2(\Gamma)} \leqslant c\|\zeta\|_{L^2(\Omega)\times L^2(\Gamma)}. \tag{6.11}$$

We write the error,

$$e = (u - u_h^\ell, v - v_h^\ell) \in L^2(\Omega) \times L^2(\Gamma),$$

as the data for the dual problem and test with $(\eta, \xi) = e$ so that

$$\|e\|^2_{L^2(\Omega)\times L^2(\Gamma)} = a(e, z_e) = F_h(z_e).$$

Hence, using (6.8) combined with the H^1 error bound (6.1) and the dual regularity result (6.11), we have

$$\|e\|^2_{L^2(\Omega)\times L^2(\Gamma)} = F_h(z_e) \leqslant C_2 h^{k+1} \|e\|_{L^2(\Omega)\times L^2(\Gamma)},$$

with C_2 as in the statement of the theorem. □

7. Numerical results

We have implemented the above finite element method using the ALBERTA finite element toolbox (Schmidt *et al.*, 2005).

The data were chosen, with $\alpha = \beta = 1$, so that the exact solution is

$$\begin{aligned} u(x_1, x_2, x_3) &= \beta \exp(-x_1(x_1 - 1)x_2(x_2 - 1)), \\ v(x_1, x_2, x_3) &= (\alpha + x_1(1 - 2x_1) + x_2(1 - 2x_2)) \exp(-x_1(x_1 - 1)x_2(x_2 - 1)). \end{aligned}$$

We calculate the right-hand side by setting $(f_h, g_h) = \widetilde{I}_h(f, g)$. We ran two simulations: one with $k = 1$, one with $k = 2$. We present the error calculated after solving the matrix system at each mesh size in Tables 1–4. A plot of the solution is provided in Fig. 5. We define the experimental order of convergence (eoc) between two errors $E(h_1)$ and $E(h_2)$ at mesh sizes h_1 and h_2 by $eoc(h_1, h_2) = \log \frac{E(h_1)}{E(h_2)} (\log \frac{h_1}{h_2})^{-1}$.

TABLE 1 *Error table for the case $k = 1$: bulk errors, $\|u - u_h\|$*

h	L^2 error	eoc	H^1 error	eoc
1.000000e+00	1.556084e−01	—	8.412952e−01	
8.201523e−01	6.945582e−02	4.068547	6.031542e−01	1.678406
4.799888e−01	2.375760e−02	2.002490	3.485974e−01	1.023385
2.555341e−01	6.692238e−03	2.009740	1.831428e−01	1.021009
1.321787e−01	1.744647e−03	2.039433	9.301660e−02	1.027742
6.736035e−02	4.427043e−04	2.034429	4.672631e−02	1.021320
3.399254e−02	1.112504e−04	2.019429	2.339324e−02	1.011617

TABLE 2 *Error table for the case $k = 1$: surface errors, $\|v - v_h\|$*

h	L^2 error	eoc	H^1 error	eoc
1.000000e+00	5.080238e−01	—	2.908569e+00	
8.201523e−01	1.591067e−01	5.855554	1.607240e+00	2.991664
4.799888e−01	4.342084e−02	2.424061	8.413412e−01	1.208220
2.555341e−01	1.108272e−02	2.166144	4.247143e−01	1.084348
1.321787e−01	2.785873e−03	2.094697	2.128454e−01	1.048012
6.736035e−02	6.973524e−04	2.054635	1.064757e−01	1.027520
3.399254e−02	1.743772e−04	2.026669	5.324210e−02	1.013381

TABLE 3 *Error table for the case $k = 2$: bulk errors, $\|u - u_h\|$*

h	L^2 error	eoc	H^1 error	eoc
1.000000e+00	3.894207e−02	–	3.511490e−01	
8.172473e−01	1.034114e−02	6.570149	1.476235e−01	4.293793
5.060717e−01	1.304277e−03	4.320133	4.026584e−02	2.710747
2.773996e−01	1.737998e−04	3.352355	1.061322e−02	2.217832
1.447909e−01	2.259868e−05	3.137667	2.723960e−03	2.091786
7.391824e−02	2.882693e−06	3.062727	6.894787e−04	2.043497

TABLE 4 *Error table for the case $k = 2$: surface errors, $\|v - v_h\|$*

h	L^2 error	eoc	H^1 error	eoc
1.000000e+00	1.538024e−01	–	1.258018e+00	
8.172473e−01	2.188515e−02	9.661695	3.745396e−01	6.003538
5.060717e−01	3.332406e−03	3.927097	1.052173e−01	2.649211
2.773996e−01	4.516347e−04	3.324205	2.718041e−02	2.251310
1.447909e−01	5.816879e−05	3.152298	6.874227e−03	2.114402
7.391824e−02	7.342240e−06	3.078402	1.725037e−03	2.056324

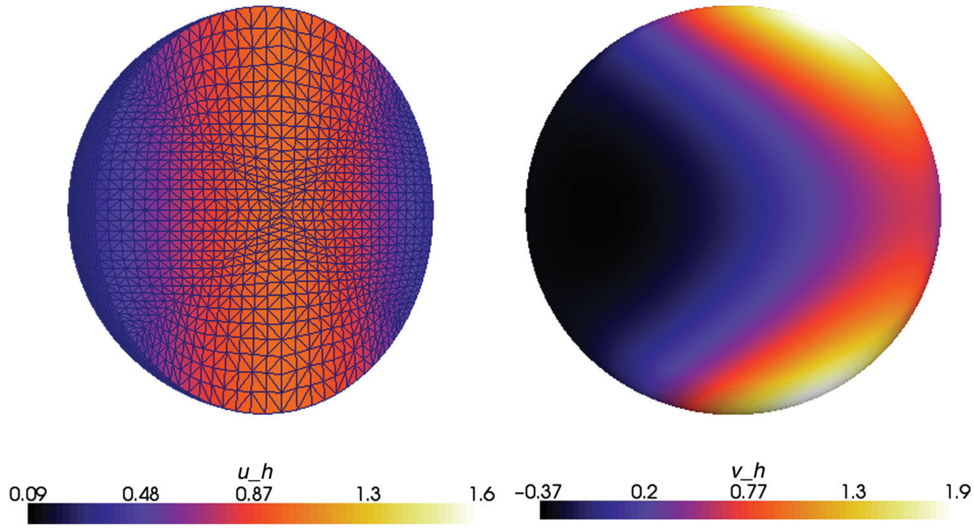

FIG. 5. Plot of the solution of the finite element scheme at $h \approx .2$, $k = 2$, along the plane $x = y$ in Ω_h, with mesh (left) and the surface Γ_h (right).

Acknowledgements

The authors thank the Centre for Scientific Computing at the University of Warwick for supplying valuable computation time.

Funding

The work of C.M.E. was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) Grant EP/G010404 and the work of T.R. was supported by a EPSRC Ph.D. studentship.

REFERENCES

ALBERTA, B., JOHNSON, A., LEWIS, J., RAFF, M., ROBERTS, K. & WALTER, P. (2002) *Molecular Biology of the Cell*, 4th edn. New York: Garland Science.

AUBIN, T. (1982) *Nonlinear Analysis on Manifolds, Monge–Ampère Equations*. New York: Springer.

BÄNSCH, E. & DECKELNICK, K. (1999) Optimal error estimates for the Stokes and Navier–Stokes equations with slip-boundary condition. *Math. Model. Numer. Anal.*, **33**, 923–938.

BARRETT, J. W. & ELLIOTT, C. M. (1987) A practical finite element approximation of a semi-definite Neumann problem on a curved domain. *Numer. Math.*, **51**, 23–36.

BERNARDI, C. (1989) Optimal finite element interpolation on curved domains. *SIAM J. Numer. Anal.*, **26**, 1212–1240.

BOOTY, M. R. & SIEGEL, M. (2010) A hybrid numerical method for interfacial fluid flow with soluble surfactant. *J. Comput. Phys.*, **229**, 3864–3883.

BRENNER, S. C. & SCOTT, L. R. (2002) *The Mathematical Theory of Finite Element Methods*. New York: Springer.

CIARLET, P. G. & RAVIART, P.-A. (1972a) Interpolation theory over curved elements, with applications to finite element methods. *Comput. Methods Appl. Mech. Eng.*, **1**, 217–249.

Ciarlet, P. G. & Raviart, P.-A. (1972b) The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. *The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations* (A. K. Aziz ed.). New York: Academic Press.

Deckelnick, K., Dziuk, G. & Elliott, C. M. (2005) Computation of geometric partial differential equations and mean curvature flow. *Acta Numer.*, **14**, 139–232.

Deckelnick, K., Günther, A. & Hinze, M. (2009) Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains. *SIAM J. Control Optim.*, **48**, 2798–2819.

Demlow, A. (2009) Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.*, **47**, 805–827.

Dubois, F. (1990) Discrete vector potential representation of a divergence-free vector field in three-dimensional domains: numerical analysis of a model problem. *SIAM J. Numer. Anal.*, **27**, 1103–1141.

Dziuk, G. (1988) Finite elements for the Beltrami operator on arbitrary surfaces. *Partial Differential Equations and Calculus of Variations* (S. Hildebrandt & R. Leis eds). Lecture Notes in Mathematics, vol. 1357. Berlin: Springer, pp. 142–155.

Dziuk, G. & Elliott, C. M. (2007a) Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27**, 262–292.

Dziuk, G. & Elliott, C. M. (2007b) Surface finite elements for parabolic equations. *J. Comput. Math.*, **25**, 385–407.

Evans, L. C. (1998) *Partial Differential Equations*. Providence, RI: American Mathematical Society.

Gilbarg, D. & Trudinger, N. S. (1983) *Elliptic Partial Differential Equations of Second Order*. Berlin: Springer.

Grisvard, P. (2011) *Elliptic Problems in Nonsmooth Domains*. Philadelphia, PA: SIAM.

Heine, C.-J. (2005) Isoparametric finite element approximation of curvature on hypersurfaces.

Hildebrant, S. (1982) *Analysis 2*. Berlin, Heidelberg, New York: Springer.

Kwon, Y.-I. & Derby, J. J. (2001) Modeling the coupled effects of interfacial and bulk phenomena during solution crystal growth. *J. Cryst. Growth*, **230**, 328–335.

Ladyzhenskaia, O. A. & Uraltseva, N. N. (1968) *Linear and Quasilinear Elliptic Equations* (Translated by Scripta Technica. Translation editor: Leon Ehrenpreis). New York: Academic Press.

Lenoir, M. (1986) Optimal isoparametric finite elements and error estimates for domains involving curved boundaries. *SIAM J. Numer. Anal.*, **23**, 562–580.

Medvedev, E. S. & Stuchebrukhov, A. A. (2011) Proton diffusion along biological membranes. *J. Phys.: Condens. Matter*, **23**, 234103.

Novak, I. L., Gao, F., Choi, Y.-S., Resasco, D., Schaff, J. C. & Slepchenko, B. M. (2007) Diffusion on a curved surface coupled to diffusion in the volume: application to cell biology. *J. Comput. Phys.*, **226**, 1271–1290.

Rätz, A. & Röger, M. (2011) Turing instabilities in a mathematical model for signaling networks. *J. Math. Biol. Online First*, 1–30.

Sbalzarini, I. F., Hayer, A., Helenius, A. & Koumoutsakos, P. (2006) Simulations of (an)isotropic diffusion on curved biological surfaces. *Biophysics J.*, **90**, 878–885.

Schmidt, A., Siebert, K. G., Köster, D. & Heine, C.-J. (2005) *Design of Adaptive Finite Element Software: The Finite Element Toolbox ALBERTA*. Berlin–Heidelberg: Springer, p. 315.

Schwartz, P., Adalsteinsson, D., Colella, P., Arkin, A. P. & Onsum, M. (2005) Numerical computation of diffusion on a surface. *Proc. Natl. Acad. Sci. USA*, **102**, 11151–11156.

Scott, L. R. (1973) Finite element techniques for curved boundaries. *Ph.D. Thesis*, Massachusetts Institute of Technology, Cambridge, MA.

Wloka, J. (1987) *Partial Differential Equations*. Cambridge, UK: Cambridge University Press, p. 532.

Zlamal, M. (1973) Curved elements in the finite element method. I. *SIAM J. Numer. Anal.*, **10**, 229–240.

Zlamal, M. (1974) Curved elements in the finite element method. II. *SIAM J. Numer. Anal.*, **11**, 347–362.