'A GENERALISED DIFFUSION EQUATION FOR PHASE SEPARATION OF A MULTI-COMPONENT MIXTURE WITH INTERFACIAL FREE ENERGY'

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IMA Preprint Series # 887

October 1991

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Acknowlegement: - This research was supported by the Institute for Mathematics and Its Applications with funds provided by the National Science Foundation whilst the authors took part in the 1990-91 Programme 'Phase Transitions and Free Boundaries'.

Abstract

A nonlinear multicomponent diffusion equation incorporating uphill diffusion and capillarity effects is studied. In the binary case the problem is the Cahn-Hilliard equation for a regular solution free energy. Global existence is proved. It is shown that the deep quench limit is a parabolic type obstacle problem.

§1 Introduction

This paper is concerned with a system of nonlinear diffusion equations modelling isothermal phase separation of an ideal mixture of $N\ (\geq 2)$ components occupying an isolated region $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$. (Morral and Cahn [1971], Kirkaldy and Young [1987], Purdy [1990]). We begin by deriving the equations in the framework of non-equilibrium thermodynamics. (c.f. de Groot and Mazur [1962], Gurtin [1988].) The basic physical quantities, defined for all $x \in \Omega$ and all time t, are the mass fraction $u_i(x,t)$, the mass flux $\vec{J}_i(x,t)$ and the chemical potential $\mu_i(x,t)$ for each component $i = 1,2.\ .\ N$ together with the total free energy $G(x,t)$. Clearly, by definition,

$$\sum_{i=1}^{N} u_i(x,t) = 1 \qquad x \in \Omega,\ t \geq 0 \tag{1-1a}$$

and

$$0 \leq u_i\ (x,t) \leq 1 \qquad x \in \Omega,\ t \geq 0 \tag{1-1b}$$

The law of mass conservation is written as, for any subregion $\mathcal{R}$ of Ω,

$$\frac{d}{dt} \int_{\mathcal{R}} u_i(x,t)\, dx = \int_{\partial\mathcal{R}} \vec{J}_i \cdot \vec{\nu}\ ds \quad \forall i \tag{1-2}$$

where $\vec{\nu}$ denotes the unit outward pointing normal. We use the notation η for N-vectors , $\vec{z}$ for d-vectors and '$\cdot$' for the scalar product of two vectors. It follows from summation of (1-2) over i that in order for (1-1a) to hold

$$\sum_{i=1}^{N} \vec{J}_i(x,t) = 0 \qquad x \in \Omega,\ t > 0 \tag{1-3}$$

The homogeneous free energy of the mixture with composition $\mathbf{u}$ is given by $\Psi\big(\mathbf{u}(x,t)\big)$ where $\Psi: \mathbb{R}_+^N \to \mathbb{R}$ is a prescribed mapping. In order to model capillarity or interfacial energy associated with large gradients of the composition we follow Cahn & Hilliard [1958] and use the gradient energy $\frac{1}{2}\,\boldsymbol{\Gamma}\,\nabla\mathbf{u}\cdot\nabla\mathbf{u}$ where $\boldsymbol{\Gamma} = \{\Gamma_{ij}\}_{i,j=1}^{N}$ is constant positive semi-definite fourth order tensor with $\Gamma_{ij}\left(=\Gamma_{ji}\right)$ being d x d matrices and

$$\left(\boldsymbol{\Gamma}\,\nabla\mathbf{u}\right)_i = \sum_{j=1}^{N}\Gamma_{ij}\,\nabla u_j \qquad ; \qquad \boldsymbol{\Gamma}\,\nabla\mathbf{u}\cdot\nabla\mathbf{v} \equiv \sum_{i,j}\Gamma_{ij}\,\nabla u_j\cdot\nabla v_i\ .$$

The total free energy is taken to be the sum of the homogeneous free energy and the gradient energy so that

$$G(x,t) := \Psi\big(\mathbf{u}(x,t)\big) + \frac{1}{2}\,\boldsymbol{\Gamma}\,\nabla\mathbf{u}\cdot\nabla\mathbf{u} \tag{1-4}$$

Thus, as in the Cahn-Hilliard model for phase separation in a binary mixture, we have a total free energy functional $\mathcal{E}(\cdot)$ given by

$$\mathcal{E}(\mathbf{u}) := \int_\Omega\Big[\Psi(\mathbf{u}) + \frac{1}{2}\,\boldsymbol{\Gamma}\,\nabla\mathbf{u}\cdot\nabla\mathbf{u}\Big]\,dx\ . \tag{1-5}$$

In the theory of multi-component diffusion without capillarity the chemical potentials for each component i is given by

$$\mu_i^{o} := \partial_i \Psi(\mathbf{u}) \tag{1-6}$$

where $\partial_i(\cdot)$ denotes the partial derivative with respect to component i . With capillarity effects the vector μ of chemical potentials is taken to be the functional derivative of $\mathcal{E}(\cdot)$ evaluated at $\mathbf{u} \in \mathbf{H}^1(\Omega)$ so that

$$\begin{aligned} \langle \mu, \eta \rangle := \; & \langle D\mathcal{E}(\mathbf{u}), \eta \rangle \\ & \qquad \forall \eta \in \mathbf{H}^1(\Omega) \qquad (1\text{-}7) \\ = \; & \left(\Gamma \nabla \mathbf{u}, \nabla \eta\right) + \left(\mu^{o}, \eta\right) . \end{aligned}$$

Formally it follows that the relationship between μ and $\mathbf{u}$ is given by the boundary value problem

$$\mu = \mu^{o} - \nabla\left(\Gamma \nabla \mathbf{u}\right) \qquad x \in \Omega,\ t > 0 \tag{1-8a}$$

$$\left(\Gamma \nabla \mathbf{u}\right)_i \cdot \vec{\nu} = 0 \qquad \forall i \qquad x \in \partial\Omega,\ t > 0 \tag{1-8b}$$

The constitutive relation for the mass fluxes is assumed to be of the isotropic form

$$\vec{J}_i := -\sum_{j=1}^{N} L_{ij} \nabla \mu_j \equiv -\left(\boldsymbol{L} \nabla \mu\right)_i \tag{1-9}$$

where $\mathbf{L}$ is a symmetric $N \times N$ matrix with constant elements L_{ij} $\left(\boldsymbol{L} = \{L_{ij}\mathbf{I}\}\right.$ is a fourth order tensor$\left.\right)$ which, for (1-3) to hold, is assumed to satisfy

$$\mathbf{L}\ \mathbf{e} = 0 \tag{1-10}$$

where $\{\mathbf{e}\}_i = 1 \ \forall i$. Thus the diffusion equations arising from the mass balance equations (1-2) become

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla\left(\boldsymbol{L}\nabla \mu\right) \qquad x \in \Omega,\ t > 0 \tag{1-11a}$$

coupled with the no mass flux boundary condition

$$\left(\boldsymbol{L}\nabla \mu\right)_i \cdot \vec{\nu} = 0 \qquad \forall i \qquad x \in \partial\Omega,\ t > 0 \tag{1-11b}$$

In order for this diffusion process to be dissipative we also assume that $\boldsymbol{L}$ is positive semi-definite. This yields the property that the total free energy functional is decreasing in time viz

$$\frac{d\mathcal{E}}{dt}(\mathbf{u}(t)) = \langle D\mathcal{E}(\mathbf{u}),\ \mathbf{u}_t \rangle = \langle \mu,\ \mathbf{u}_t \rangle$$

$$= \left(\mu\ ,\ \nabla \boldsymbol{L} \nabla \mu\right) = \left(-\boldsymbol{L}\nabla\mu, \nabla\mu\right) \le 0\ .$$

Furthermore the following version of the second law of thermodynamics

$$\frac{d}{dt}\int_{\mathcal{R}} G(x,t)\,dx \;+\; \int_{\partial\mathcal{R}} \left[\mu \cdot \mathbf{J}_\nu - \mathbf{u}_t \cdot (\boldsymbol{\Gamma}\nabla\mathbf{u})_\nu\right] ds \;\le\; 0 \qquad (1\text{-}12)$$

is satisfied for each subregion $\mathcal{R}$ of Ω, where we have set $\{\mathbf{J}_\nu\}_i = \vec{\mathbf{J}}_i \cdot \vec{\nu}$ and $\{(\boldsymbol{\Gamma}\nabla\mathbf{u})_\nu\}_i = \sum_{j=1}^{d} \boldsymbol{\Gamma}_{ij}\nabla\mathbf{u}_j \cdot \vec{\nu}$. Inequality (1-12) is a generalisation to multi-component diffusion with capillarity of the Clausius-Duhem inequality for binary diffusion with capillarity given by Gurtin [1988]. To see that (1-12) holds, observe that the left hand side can be rewritten using (1.4), (1.8) and integration by parts as

$$\int_{\mathcal{R}} \mathbf{u}_t \cdot \mu \;+\; \int_{\partial\mathcal{R}} \mu \cdot \mathbf{J}_\nu$$

and using (1.9), (1.11a) and an integration by parts we are left with

$$-\int_{\mathcal{R}} \boldsymbol{L}\nabla\mu \cdot \nabla\mu\,dx \;.$$

Thus the constitutive assumption that $\boldsymbol{L}$ is positive semi-definite yields the desired inequality.

We now make further constitutive assumptions. First we assume a 'regular solution' for the homogeneous free energy: -

$$\Psi(\mathbf{u}) := \theta\sum_{i=1}^{N} u_i \ln u_i \;-\; \tfrac{1}{2}\,\mathbf{u}^T A\mathbf{u} \qquad (1\text{-}13)$$

where θ is the absolute temperature and A is a constant symmetric $N \times N$ matrix with largest eigenvalue $\lambda_A > 0$. Here we have taken the Boltzman constant to be 1 so temperature is scaled accordingly. It follows that there exists a critical temperature θ_c so that for θ greater (lesser) than θ_c the homogeneous free energy $\Psi(\cdot)$ is convex (non-convex).

Second we assume that $\boldsymbol{\Gamma}$ is $\gamma \boldsymbol{I}$ so that

$$\mathcal{E}(\mathbf{u}) := \int_{\Omega} \left[\Psi(\mathbf{u}) + \frac{\gamma}{2} |\nabla \mathbf{u}|^2 \right] dx. \qquad (1\text{-}14)$$

Third we assume that $\boldsymbol{L}$ is constant, that the kernel of $\boldsymbol{L}$ is one-dimensional and that

$$\mathbf{L}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \geq \ell_0 \mathbf{P}\boldsymbol{\eta} \cdot \mathbf{P}\boldsymbol{\eta} \qquad (1\text{-}15)$$

where

$$\mathbf{P}\boldsymbol{\eta} := \boldsymbol{\eta} - \mathbf{e}\sum \boldsymbol{\eta} \quad ; \qquad \sum \boldsymbol{\eta} = \frac{1}{N} \sum_{i=1}^{N} \eta_i .$$

It is convenient to introduce the vector of generalised chemical potential differences $\mathbf{w}$ defined by

$$\mathbf{w} := \mathbf{P} \boldsymbol{\mu} \; . \qquad (1\text{-}16)$$

The equations (1-7) and (1-10) become

$$\mathbf{w} = \mathbf{P}(\theta \boldsymbol{\phi}(\mathbf{u}) - A\mathbf{u}) - \gamma \Delta \mathbf{u} \qquad x \in \Omega, \ t > 0 \qquad (1\text{-}17a)$$

$$\gamma \frac{\partial \mathbf{u}}{\partial \nu} = 0 \qquad x \in \partial\Omega, \ t > 0 \qquad (1\text{-}17b)$$

where $\{\boldsymbol{\phi}(\mathbf{u})\}_i := \phi(u_i) \equiv \psi'(u_i) - 1 ; \quad \psi(r) := r \ln r ,$

and
$$\frac{\partial \mathbf{u}}{\partial t} = \nabla \left(\boldsymbol{L} \nabla \mathbf{w} \right) \qquad x \in \Omega, \ t > 0 \qquad (1\text{-}18a)$$

$$\left(\boldsymbol{L} \nabla \mathbf{w} \right)_\nu = 0 \qquad x \in \partial\Omega, \ t > 0 \qquad (1\text{-}18b)$$

Here we have used the facts $\sum \mathbf{w} = 0 = \sum \mathbf{u} - 1/_N$.

The principal result of this paper is an existence theorem for the system (1-17, 1-18) with the initial condition

$$\mathbf{u}(x, 0) = \mathbf{u}_o . \qquad (1\text{-}19)$$

The major difficulty is that $\phi(r)$ is singular at $r = 0$ and (1-17) can have no meaning if $u_i = 0$ in an open set of non-zero measure. Also there is no maximum principle which precludes this. However it is precisely this form of $\phi(\cdot)$ that maintains the constraint (1-1b) on the composition. Our result is stated as follows. We use the notation $⨍ \eta = \int_\Omega \eta \, dx / |\Omega|$.

Theorem 1

Let $T > 0$ and $\mathbf{u}_o \in K = \left\{ \eta \in \mathbf{H}^1(\Omega) : \sum \eta = 1/_N , \ \eta \geq 0 \right\}$.

Suppose that $\delta \mathbf{e} < \fint \mathbf{u}_o < (1-\delta)\mathbf{e}$ then there exists a unique pair $\{\mathbf{u}, \mathbf{w}\}$ such that

$$\mathbf{u} \in C\left[0,T; \left(\mathbf{H}^1(\Omega)\right)'\right] \cap L^\infty\left(0,T; \mathbf{H}^1(\Omega)\right)$$

$$\frac{d\mathbf{u}}{dt} \in L^2\left(0,T; \left(\mathbf{H}^1(\Omega)\right)'\right) , \quad \sqrt{t}\ \frac{d\mathbf{u}}{dt} \in L^2\left(0,T; \mathbf{H}^1(\Omega)\right)$$

$$\bar{\mathbf{w}} = \mathbf{w} - \fint \mathbf{w} \in L^2\left(0,T; \mathbf{H}^1(\Omega)\right)$$

$$\sqrt{t}\ \mathbf{w} \in L^\infty\left(0,T; \mathbf{H}^1(\Omega)\right)$$

$$\sqrt{t}\,\theta\,\phi(\mathbf{u}) \in L^\infty\left(0,T; L^2(\Omega)\right)$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_o ,$$

$$\mathbf{u}(\cdot,t) \in K \quad \forall\, t > 0 \quad , \quad \fint \mathbf{u}(\cdot,t) = \fint \mathbf{u}_o$$

and for all $\xi \in C[0,T]$ and $\eta \in \mathbf{H}^1(\Omega)$

$$\int_o^T \xi(t)\left\{\frac{d}{dt}\langle \mathbf{u}, \eta\rangle + \left(\boldsymbol{L}\nabla\bar{\mathbf{w}}, \nabla\eta\right)\right\} dt = 0 \qquad (1\text{-}20a)$$

$$\int^T \xi(t)\left\{\left(\mathbf{w} - \theta\,\phi(\mathbf{u}) + A\mathbf{u} - \mathbf{e}\sum\left(\theta\,\phi(\mathbf{u}) - A\mathbf{u}\right), \eta\right) - \gamma\left(\nabla\mathbf{u}, \nabla\eta\right)\right\} dt = 0 \quad \square \qquad (1\text{-}20b)$$

Based upon this existence theorem it is possible to justify the deep quench limit problem $\theta \to 0$ studied by Blowey and Elliott [1991a,b] for binary diffusion with capillarity. See also Oono and Puri [1988].

Theorem 2

Let $T > 0$ and $\mathbf{u}_O \in K$. There exists a unique pair $\{\mathbf{u}, \mathbf{w}\}$ such that

$$\mathbf{u} \in C\left[0, T; \left(\mathbf{H}^1(\Omega)\right)'\right] \cap L^\infty\left(0, T; \mathbf{H}^1(\Omega)\right)$$

$$\frac{d\mathbf{u}}{dt} \in L^2\left(0, T; \left(\mathbf{H}^1(\Omega)\right)'\right) \quad , \quad \sqrt{t}\,\frac{d\mathbf{u}}{dt} \in L^2\left(0, T; \mathbf{H}^1(\Omega)\right)$$

$$\bar{\mathbf{w}} := \mathbf{w} - \fint \mathbf{w} \in L^2\left(0, T; \mathbf{H}^1(\Omega)\right) \quad , \quad \sqrt{t}\,\mathbf{w} \in L^\infty\left(0, T; \mathbf{H}^1(\Omega)\right)$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_O$$

$$\mathbf{u}(\cdot, t) \in K \qquad \forall t > 0$$

and for $\xi \in C\,[0, T]$ and $\eta \in \mathbf{H}^1(\Omega)$

$$\int_O^T \xi(t) \left\{ \frac{d}{dt} < \mathbf{u}, \boldsymbol{\eta} > + \left(\boldsymbol{L} \nabla \bar{\mathbf{w}}, \nabla \boldsymbol{\eta} \right) \right\} dt = 0 \tag{1-21a}$$

and for $\xi(\geq 0) \in C\,[0, T)$ and $\boldsymbol{\eta} \in K$

$$\int_O^T \xi(t) \left\{ \gamma\left(\nabla \mathbf{u}, \nabla \boldsymbol{\eta} - \nabla \mathbf{u}\right) - \left(A\mathbf{u} - \mathbf{e} \sum A\mathbf{u} + \mathbf{w}, \boldsymbol{\eta} - \mathbf{u}\right) \right\} dt \geq 0 \qquad \square \tag{1-21b}$$

The layout of the paper is as follows. In Section 2 an approximation to (1-10) is studied. Using estimates derived in Section 2, Theorem 1 is proved in Section 3. The proof of Theorem 2 is given in Section 4.

§2 A regularised problem

We shall consider a family of regularised problems parameterised by ε and obtain the existence result by passing to the limit $\varepsilon = 0$. For each ε small and positive we define

$$\phi_\varepsilon(r) = \begin{cases} \ln r & r \geq \varepsilon \\ \left(\ln\varepsilon - 1 + \frac{r}{\varepsilon} \right) & r < \varepsilon \end{cases} \tag{2-1}$$

We set

$$\boldsymbol{\phi}^\varepsilon := \boldsymbol{\phi}^\varepsilon(\mathbf{u}^\varepsilon) \equiv \left\{ \phi_\varepsilon(u_i^\varepsilon) \right\}_i$$

$$\mathbf{q}^\varepsilon = A\,\mathbf{u}^\varepsilon$$

The regularised equations are:

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial t} = \nabla\left(\boldsymbol{L} \nabla \mathbf{w}^\varepsilon \right) \tag{2-2a}$$

$$\mathbf{w}^\varepsilon = -\gamma\,\Delta\,\mathbf{u}^\varepsilon + \theta\,\boldsymbol{\phi}^\varepsilon - \mathbf{q}^\varepsilon + \mathbf{e} \sum \left(\mathbf{q}^\varepsilon - \theta\,\boldsymbol{\phi}^\varepsilon \right) \tag{2-2b}$$

holding in Ω for $t > 0$, together with the boundary conditions on $\partial\Omega$

$$\left(\boldsymbol{L} \nabla \mathbf{w}^\varepsilon \right)_\nu = 0 \qquad \frac{\partial \mathbf{u}^\varepsilon}{\partial \nu} = 0 \tag{2-2c}$$

and initial condition

$$\mathbf{u}^{\varepsilon}(\cdot,0) = \mathbf{u}_0 \quad . \tag{2.2d}$$

By using standard arguments based on Galerkin approximations it is easy to show that (2.2) possesses a pair of solutions $\{\mathbf{u}^{\varepsilon}, \mathbf{w}^{\varepsilon}\}$ such that for each $T > 0$,

$$\mathbf{u}^{\varepsilon} \in L^{\infty}\left(0,T;\ \mathbf{H}^1(\Omega)\right), \quad d\mathbf{u}^{\varepsilon}/dt \in L^2\left(0,T;(\mathbf{H}^1(\Omega))'\right)$$

$$\mathbf{w}^{\varepsilon} \in L^2\left(0,T;\ \mathbf{H}^1(\Omega)\right)$$

and for a.e. $t \in (0,T)$ equations (2-2a,b,c) hold in the following weak sense: for all $\eta \in \mathbf{H}^1(\Omega)$

$$\frac{d}{dt}\left\langle \mathbf{u}^{\varepsilon}, \eta \right\rangle + \left(\boldsymbol{L}\, \nabla \mathbf{w}^{\varepsilon}, \nabla \eta\right) = 0 \tag{2-3a}$$

$$\left(\mathbf{w}^{\varepsilon}, \eta\right) = \gamma\left(\nabla \mathbf{u}^{\varepsilon}, \nabla \eta\right) + \left(\theta\, \boldsymbol{\phi}^{\varepsilon} - \mathbf{q}^{\varepsilon} - \mathbf{e} \sum\left(\theta\, \boldsymbol{\phi}^{\varepsilon} - \mathbf{q}^{\varepsilon}\right), \eta\right). \tag{2-3b}$$

For our purposes we wish to obtain sufficient estimates independent of ε in order to pass to the limit.

We define a regularised homogeneous free energy by

$$\psi^{\varepsilon}(r) := \begin{cases} r \ln r & r \geq \varepsilon \\ \left(\dfrac{r^2}{2\varepsilon} + r \ln \varepsilon - \dfrac{\varepsilon}{2}\right) & r < \varepsilon \end{cases} \tag{2-4}$$

and $\Psi^{\varepsilon} : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\Psi^{\varepsilon}(\mathbf{r}) = \theta \sum_{i=1}^{N} \psi^{\varepsilon}(r_i) - \frac{1}{2} \mathbf{r}^T A \mathbf{r} \qquad (2\text{-}5)$$

Lemma 2-1

There exists an $\varepsilon_o > 0$ and $k > 0$ such that for all $\varepsilon < \varepsilon_o$

$$\Psi^{\varepsilon}(\mathbf{r}) \geq -k \quad \forall\ \mathbf{r} \in \mathbb{R}^N \text{ such that } \sum \mathbf{r} = {}^1/_N \qquad (2\text{-}6)$$

Proof

Observe that

$$\Psi^{\varepsilon}(\mathbf{r}) \geq \sum_{i=1}^{N} \left[\theta\ \psi^{\varepsilon}(r_i) - \lambda_A r_i^2 \tfrac{1}{2} \right] \quad \forall\ \mathbf{r} \in \mathbb{R}^N$$

Since for $\varepsilon_o < {}^1/_e$,

$$\psi^{\varepsilon}(r) \geq -{}^1/_e$$

we need only consider estmating $\Psi^{\varepsilon}(\mathbf{r})$ from below for $\max_i |r_i| > 1$.
Set

$$R_m = \min_j r_j \quad , \quad R_M = \max_j r_j \ .$$

It follows that

$$1-(N-1)R_M \le R_m \le \frac{1-R_M}{(N-1)}$$

and

$$\Psi^{\varepsilon}(r) \ge -\theta(N-1)\cdot/e + \theta\left(\frac{R_m^2}{2\varepsilon} + R_m \ln\varepsilon - \frac{\varepsilon}{2}\right) - N\lambda_A(N-1)^2 R_M^2/2.$$

Choosing ε_0 sufficiently small (depending on θ, N and λ_A) gives the result. □

In the next proposition we show that (2.2) possesses natural mass conservation and energy decay properties. We introduce the total regularised energy by

$$\mathcal{E}^{\varepsilon}(\mathbf{v}) := \int_{\Omega} \left[\frac{\gamma}{2} |\nabla \mathbf{v}|^2 + \Psi^{\varepsilon}(\mathbf{v}) \right] dx \qquad (2\text{-}7)$$

Proposition 2-1

a) *Conservation of Mass*

$$⨍ \mathbf{u}^{\varepsilon}(.,t) = ⨍ \mathbf{u}_0 \qquad (2\text{-}8a)$$

b) *Conservation of Total Local Mass*

$$\sum \mathbf{u}^{\varepsilon}(\mathbf{x},t) = {}^{1}/_{N} \qquad x \in \Omega, \ t > 0 \qquad (2\text{-}8b)$$

c) *Energy Decay*

$$\frac{d\,\mathcal{E}^{\varepsilon}(\mathbf{u}^{\varepsilon})}{dt} + \int_{\Omega} \boldsymbol{L}\,\nabla \mathbf{w}^{\varepsilon} \cdot \nabla \mathbf{w}^{\varepsilon}\, dx = 0. \qquad (2\text{-}8c)$$

d) *Conservation of Total Chemical Potential*

$$\sum \mathbf{w}^{\varepsilon}(x,t) = 0 \qquad (2\text{-}8d)$$

Proof

a) Taking $\boldsymbol{\eta} = \mathbf{e}_k = \{\delta_{ik}\}_i$ for each k yields (2-8a) immediately.

b) Setting

$$U^{\varepsilon} = \sum_{i=1}^{N} u_i^{\varepsilon}, \qquad W^{\varepsilon} = \sum_{i=1}^{N} w_i^{\varepsilon}$$

and taking $\boldsymbol{\eta} = \eta\,\mathbf{e}_k$ (k=1,..N) with $\eta \in H^1(\Omega)$ in (2-2a,b) we obtain after summing,

$$\left\langle \frac{dU^{\varepsilon}}{dt}, \eta \right\rangle + \left(\mathbf{L}\,\nabla W^{\varepsilon}, \nabla\eta \right) = 0$$

$$\left(W^{\varepsilon}, \eta \right) + \gamma \left(\nabla U^{\varepsilon}, \nabla\eta \right) = 0$$

Since

$$U^{\varepsilon}(.,0) = \sum_{i=1}^{N} u_i(.,0) = 1$$

we find that these linear equations have the unique solution

$$U^{\varepsilon}(x,t) \equiv 1, \quad W^{\varepsilon}(x,t) \equiv 0$$

which implies (2-8b).

c) By differentiating (2-7) with respect to t we find that the regularised energy satisfies

$$\frac{d\mathcal{E}^{\varepsilon}}{dt}(\mathbf{u}^{\varepsilon}) = \left(-\gamma \, \Delta \, \mathbf{u}^{\varepsilon} + \theta \, \boldsymbol{\phi}^{\varepsilon}(\mathbf{u}^{\varepsilon}) - \mathbf{q}^{\varepsilon}, \frac{\partial \mathbf{u}^{\varepsilon}}{\partial t} \right)$$

$$= \left(\mathbf{w}^{\varepsilon} + \mathbf{e} \left(\sum (\theta \, \boldsymbol{\phi}^{\varepsilon}(\mathbf{u}^{\varepsilon}) - \mathbf{q}^{\varepsilon} \right), \frac{\partial \mathbf{u}^{\varepsilon}}{\partial t} \right)$$

$$= \left(\mathbf{w}^{\varepsilon}, \frac{\partial \mathbf{u}}{\partial t} \right) + \left(\sum (\theta \, \boldsymbol{\phi}^{\varepsilon} - \mathbf{q}^{\varepsilon}), \frac{\partial U^{\varepsilon}}{\partial t} \right) .$$

Since $U^{\varepsilon} \equiv 1$ and (2-3a) holds we finally obtain (2-8c). □

Proposition 2-2

There exist constants $C_j (j=1,2,3)$ depending only on the initial data and independent of ε so that

$$\| \nabla \mathbf{u}^{\varepsilon}(t) \|^2 \quad + \quad \int_0^t \| \nabla \mathbf{w}^{\varepsilon} \|^2 \, d\tau \quad \le \quad C_1 \tag{2-9a}$$

$$\| \mathbf{u}^{\varepsilon} (t) \|_1 \quad \le \quad C_2 \tag{2-9b}$$

$$\sum_{i=1}^{N} \left\{ ⨍ \left[- u_i^{\varepsilon} \right]_+ \quad + \quad ⨍ \left[u_i^{\varepsilon} - 1 \right]_+ \right\} \le C_3 \Big/ \left(\theta \, | \ln \varepsilon | \right) \tag{2-9c}$$

Proof

These estimates are consequences of the fact that $\mathcal{E}^{\varepsilon}(.)$ is a Lyapunov functional for the system. Integrating (2-8c) with respect to t and using (1-15) yields

$$\gamma \, \| \nabla \mathbf{u}^{\varepsilon}(t) \|^2 + \ell_0 \int_0^t \| \nabla \mathbf{w}^{\varepsilon}(\tau) \|^2 \, d\tau + \int_{\Omega} \Psi^{\varepsilon} (\mathbf{u}^{\varepsilon}(t) \, dx \le \mathcal{E}^{\varepsilon}(\mathbf{u}_0) . \tag{2-10}$$

Inequality (2-9a) follows from Lemma 2-1 and the fact that, since $\{\mathbf{u}_0\}_i \in [0,1]$,

$$\int_{\Omega} \Psi^{\varepsilon} (\mathbf{u}_0) \quad \le \quad - \frac{1}{2} \, (A\mathbf{u}_0 , \, \mathbf{u}_0) \quad .$$

Noting (2-8a) we obtain (2-9b) by a direct use of Poincarè's inequality.

Turning to (2-9c), we first observe that (2-9b) implies that

$$(\mathbf{Au}^\varepsilon, \mathbf{u}^\varepsilon) \leq C \quad \forall t .$$

Since

$$\int_\Omega \psi^\varepsilon(u_i^\varepsilon)\, dx \geq -\theta|\Omega|/e + \theta \int_{[u_i^\varepsilon < \varepsilon]} \psi_\varepsilon(u_i^\varepsilon)\, dx$$

$$\geq -\theta|\Omega|/e + \theta \ln\varepsilon \int_{[u_i^\varepsilon < 0]} u_i^\varepsilon\, dx + \theta\varepsilon \ln\varepsilon\, |\Omega| - \theta\frac{\varepsilon}{2}|\Omega| ,$$

it follows from the inequality

$$\int_\Omega \Psi^\varepsilon(\mathbf{u}^\varepsilon)\, dx < C$$

that
$$\sum_{i=1}^N \int \left[- u_i^\varepsilon(\cdot\, , t) \right]_+ \leq C \big/ \left(\theta\, |\ln\varepsilon| \right)$$

for $\varepsilon < \varepsilon_0$ sufficiently small.

Finally we have that, using (2-8b),

$$\fint [u_i^\varepsilon - 1]_+ \quad = \quad \frac{-1}{|\Omega|} \int_{[u_i^\varepsilon > 1]} \sum_{j \neq i} u_j^\varepsilon \, (x,t) \, dx$$

$$\leq \quad \frac{1}{|\Omega|} \int_{[u_i^\varepsilon > 1]} \sum_{j \neq i} [-u_j^\varepsilon]_+ \quad dx$$

$$\leq \quad \sum_{j \neq i} \fint [-u_j^\varepsilon]_+ \quad .$$

□

Proposition 2-3

There exist constants C_4 and C_5 depending on the initial data and T such that

$$t \, \| \nabla \mathbf{w}^\varepsilon (t) \|^2 \; + \; \int_0^T s \, \| \nabla \frac{d\mathbf{u}^\varepsilon}{dt} \|^2 \, ds \; \leq \; C_4 \qquad (2\text{-}11)$$

$$\theta \, t \, \| \left(\phi^\varepsilon - \fint \phi^\varepsilon \right) - \mathbf{e} \left(\sum \phi^\varepsilon \; - \fint \sum \phi^\varepsilon \right) \|^2 \; \leq \; C_5 \qquad (2\text{-}12)$$

Proof

Differentiating (2-3b) with respect to t and taking $\eta = \frac{d\mathbf{u}^\varepsilon}{dt}$ yields

$$\left(\frac{d\mathbf{w}^{\varepsilon}}{dt}, \frac{d\mathbf{u}^{\varepsilon}}{dt} \right) = \gamma \| \nabla \frac{d\mathbf{u}^{\varepsilon}}{dt} \|^2 + \left(D(\mathbf{u}^{\varepsilon}) \frac{d\mathbf{u}^{\varepsilon}}{dt}, \frac{d\mathbf{u}^{\varepsilon}}{dt} \right)$$

$$- \left(A \frac{d\mathbf{u}^{\varepsilon}}{dt}, \frac{d\mathbf{u}^{\varepsilon}}{dt} \right) + \frac{d}{dt} \left(\left(\mathbf{q}^{\varepsilon} - \theta \boldsymbol{\phi}^{\varepsilon} \right), \frac{dU^{\varepsilon}}{dt} \right),$$

where $D(\mathbf{u}^{\varepsilon})$ is the diagonal matrix with entry $\left\{ \theta \phi_{\varepsilon}' (u_i^{\varepsilon}) \right\}$.

Since $\phi_{\varepsilon}'(\cdot) \geq 0$ and $U^{\varepsilon}(x,t) = 1$, it follows from the above equation that

$$\frac{1}{2} \frac{d}{dt} \| \nabla \mathbf{w}^{\varepsilon}(t) \|^2 + \gamma \| \nabla \frac{d\mathbf{u}^{\varepsilon}}{dt} \|^2 \leq \left(A \frac{d\mathbf{u}^{\varepsilon}}{dt}, \frac{d\mathbf{u}^{\varepsilon}}{dt} \right) .$$

Since taking $\eta = A \frac{d\mathbf{u}^{\varepsilon}}{dt}$ in (2-3a) yields

$$\left(A \frac{d\mathbf{u}^{\varepsilon}}{dt}, \frac{d\mathbf{u}^{\varepsilon}}{dt} \right) = \left(- \boldsymbol{L} \nabla \mathbf{w}^{\varepsilon}, \nabla A \frac{d\mathbf{u}^{\varepsilon}}{dt} \right)$$

$$\leq C_{\mathbf{A}} \| \boldsymbol{L} \| \, \| \nabla \mathbf{w}^{\varepsilon} \| \, \| \nabla \frac{d\mathbf{u}^{\varepsilon}}{dt} \| ,$$

we obtain after multiplying by t that

$$\frac{d}{dt} \left[t \| \nabla \mathbf{w}^{\varepsilon} \|^2 \right] + t \| \nabla \frac{d\mathbf{u}^{\varepsilon}}{dt} \|^2 \leq C(t+1) \| \nabla \mathbf{w}^{\varepsilon} \|^2 .$$

Inequality (2-11) now follows after integrating with respect to t and noting (2-9a).

Turning to the proof of estimate (2-12) we set

$\mathbf{g}^{\varepsilon} = \phi^{\varepsilon} - \left(\sum \phi^{\varepsilon}\right)\mathbf{e}$ and take $\eta = \mathbf{g}^{\varepsilon} - ⨍\mathbf{g}^{\varepsilon}$ in (2-3b) yielding

$$\theta \,\|\mathbf{g}^{\varepsilon} - ⨍\mathbf{g}^{\varepsilon}\|^2 + \gamma(\nabla \mathbf{u}^{\varepsilon}, \nabla \phi^{\varepsilon})$$

$$= \left(\mathbf{w}^{\varepsilon} - ⨍\mathbf{w}^{\varepsilon}, \mathbf{g}^{\varepsilon} - ⨍\mathbf{g}^{\varepsilon}\right) + \left(\mathbf{q}^{\varepsilon} - \sum \mathbf{q}^{\varepsilon}\mathbf{e}, \mathbf{g}^{\varepsilon} - ⨍\mathbf{g}^{\varepsilon}\right)$$

$$+ \gamma\left(\mathbf{e}\nabla U^{\varepsilon}, \nabla \phi^{\varepsilon}\right)/N$$

Therefore it holds that

$$\theta \,\|\mathbf{g}^{\varepsilon} - ⨍\mathbf{g}^{\varepsilon}\|^2 \le C\left(\|\mathbf{w}^{\varepsilon} - ⨍\mathbf{w}^{\varepsilon}\|^2 + \|\mathbf{q}^{\varepsilon} - ⨍\mathbf{q}^{\varepsilon}\|^2\right)$$

and the estimates (2-9b) and (2-11) together with the Poincaré inequality imply (2-12). □

We are now in a position to state the crucial estimate which will allow us to pass to the limit.

Proposition 2-4

There exists a constant C_6 depending on T, the initial data and θ such that for ε_0 sufficiently small

$$\|\phi^{\varepsilon}\|^2 \le C_6\, t^{-1} . \tag{2.13}$$

Proof

Recall that there exists $\delta \in (0,1)$ such that for each $i \in [1,N]$

$$\delta < \fint u_i^{\varepsilon} < 1-\delta . \tag{2-14}$$

Our estimates will be independent of ε but will depend on δ and θ; in particular they require δ and θ to be positive. We shall fix $t > 0$ and suppress the dependence on t in the following.

Set

$$\Omega^{\varepsilon} := \left\{ x \in \Omega : \max_{1 \le i \le N} u_i^{\varepsilon} > 1 + \sqrt{\frac{C_3}{\theta |\ln\varepsilon|}} \right\} \tag{2-15}$$

It follows from (2-9c) that

$$\left(\frac{C_3}{\theta |\ln\varepsilon|} \right)^{1/2} |\Omega^{\varepsilon}| < \int_{\Omega^{\varepsilon}} \left(\max_{1 \le i \le N} u_i^{\varepsilon} - 1 \right) dx$$

$$\le \sum_{i=1}^{N} \int_{\Omega} \left[u_i^{\varepsilon} - 1 \right]_+ dx$$

$$< N \frac{C_3 |\Omega|}{\theta |\ln\varepsilon|}$$

and we have

$$|\Omega^{\varepsilon}| < K_1 \frac{|\Omega|}{(\theta\,|\ln\varepsilon|)^{1/2}} \tag{2-16}$$

Set

$$\Omega_i^{\varepsilon} := \left\{ x \in \Omega \;:\; u_i^{\varepsilon} > \frac{\delta}{2} \right\} \setminus \overline{\Omega^{\varepsilon}} \tag{2-17}$$

and assume that ε_0 is sufficiently small so

$$\frac{C_3}{\theta|\ln\varepsilon|} < \frac{\delta}{4} \;. \tag{2.18}$$

Noting (2-14), (2-9c) and (2-18) we find that

$$⨍ \min\left\{u_i^{\varepsilon}, 1\right\} = ⨍ u_i^{\varepsilon} - ⨍ \left[u_i^{\varepsilon} - 1\right]_+$$

$$> \delta - \frac{C_3}{\theta|\ln\varepsilon|} > \frac{3}{4}\,\delta \qquad .$$

But also, setting

$$A_1^{\varepsilon} := \Omega_i^{\varepsilon}\;,\quad A_2^{\varepsilon} := \Omega^{\varepsilon} \cap \left[u_i^{\varepsilon} > \frac{\delta}{2}\right],$$

$$A_3^{\varepsilon} := \left[u_i^{\varepsilon} < \frac{\delta}{2}\right]$$

we find that

$$\fint \min\left\{u_i^{\varepsilon}, 1\right\} < \frac{|A_1^{\varepsilon}|}{|\Omega|} + \frac{|A_2^{\varepsilon}|}{|\Omega|} + \frac{\delta}{2}\,\frac{|A_3^{\varepsilon}|}{|\Omega|}$$

$$\le \frac{|\Omega_i^{\varepsilon}|}{|\Omega|} + \frac{|\Omega^{\varepsilon}|}{|\Omega|} + \frac{\delta}{2} .$$

The above inequalities together with (2-16) imply that

$$\frac{|\Omega_i^{\varepsilon}|}{|\Omega|} > \frac{\delta}{8} \qquad (2\text{-}19)$$

provided that ε_0 is sufficiently small so that

$$\frac{K_1}{(\theta|\ln\varepsilon|)^{1/2}} < \frac{\delta}{8}$$

Since $\phi_\varepsilon(\cdot)$ is monotone increasing we have that, using (2-18),

$$\phi_\varepsilon(u_i^{\varepsilon}) \le \phi_\varepsilon\left(\max_{1\le j\le N} u_j^{\varepsilon}\right) \le \phi_\varepsilon\left(1 + \left(\frac{C_3}{\theta|\ln\varepsilon|}\right)^{1/2}\right)$$

$$\le \ln\left(1+\delta^{1/2}\right) + 1$$

on the complement of Ω^{ε} . It follows that on Ω_i^{ε} ,

$$g_i^\varepsilon := \phi_\varepsilon(u_i^\varepsilon) - \sum \phi_\varepsilon > \ln\left(\frac{\delta}{2}\right) - \ln\left(1+\delta^{1/2}\right). \qquad (2\text{-}21)$$

Let $z_j^\varepsilon := \fint g_j^\varepsilon$. If $z_i^\varepsilon < 0$ then from (2-12) and (2-21)

$$t^{-1} C_5 > \int_{\Omega_i^\varepsilon} \left(g_i^\varepsilon - z_i^\varepsilon\right)^2 dx \geq |\Omega_i^\varepsilon| \, z_i^{\varepsilon\,2} - 2 z_i^\varepsilon \int_{\Omega_i} g_i^\varepsilon$$

$$> |\Omega_i^\varepsilon| \left(z_i^\varepsilon + 2 z_i^\varepsilon \ln \left(\frac{2+2\delta^{1/2}}{\delta}\right)\right)$$

and this implies

$$|z_i^\varepsilon|^2 < K_2 \, t^{-1} , \qquad (2\text{-}22)$$

where K_2 depends on δ . If $z_i^\varepsilon > 0$ then

$$0 = \fint \sum_{j=1}^{N} g_j^\varepsilon \quad \text{implies that}$$

$$0 < z_i^\varepsilon = \sum_{j\neq i} \fint g_j^\varepsilon < -\sum_{\{j:\, z_j < 0\}} z_j$$

and by (2-22),

$$|z_i^\varepsilon|^2 < (N-1)^2 K_2 t^{-1} .$$

Thus we have shown the existence of K_3 such that

$$|\fint g_i^{\varepsilon}|^2 < K_3 t^{-1} \quad i=1,2,\ldots N \ . \tag{2-23}$$

It follows from (2-12) that

$$\int_{\Omega} g_i^{\varepsilon\,2}\, dx \le C_S\, t^{-1} + |\Omega| \left(\fint g_i^{\varepsilon}\right)^2$$

$$\le K_4\, t^{-1} \ . \tag{2-24}$$

Set

$$\tilde{\Omega}_i^{\varepsilon} := \left\{ x \in \Omega \ : \ u_i^{\varepsilon} = \max_{1 \le j \le N} u_j^{\varepsilon} \right\} \ . \tag{2-25}$$

Since $\phi_{\varepsilon}(\cdot)$ is monotone we have that on $\tilde{\Omega}_i^{\varepsilon}$,

$$g_i \ge 0 \quad \text{and} \quad \phi_{\varepsilon}(u_i^{\varepsilon}) \ge \phi_{\varepsilon}\left(\tfrac{1}{N}\right)$$

so

$$g_i \ge \left[\phi_{\varepsilon}\left(\tfrac{1}{N}\right) - \sum \phi^{\varepsilon}\right]_+ \quad \text{on} \quad \tilde{\Omega}_i^{\varepsilon} ,$$

which yields

$$\int_{\Omega} (g_i^{\varepsilon})^2 \, dx \;\ge\; \int_{\hat{\Omega}_i^{\varepsilon}} (g_i^{\varepsilon})^2 \, dx \;\ge\; \int_{\hat{\Omega}_i^{\varepsilon}} \Big[\phi_\varepsilon\big(\tfrac{1}{N}\big) - \sum \phi^{\varepsilon}\Big]_+^2$$

and summing this inequality over $i=1,2,\dots N$, using (2-24),

$$\int_{\Omega} \Big[\phi_\varepsilon\big(\tfrac{1}{N}\big) - \sum \phi^{\varepsilon}\Big]_+^2 \, dx \;\le\; \sum_{i=1}^{N} \int_{\Omega} (g_i^{\varepsilon})^2 \, dx \;\le\; K_S \, t^{-1} . \qquad \text{(2-26)}$$

Furthermore we have for each $x \in \Omega$,

$$\max_{1\le j\le N} g_i^{\varepsilon} = \phi_\varepsilon \Big(\max_{1\le j\le N} u_j^{\varepsilon} \Big) - \frac{1}{N} \sum_{j=1}^{N} \phi_\varepsilon \big(u_j^{\varepsilon}\big)$$

$$= \frac{N-1}{N} \, \phi_\varepsilon \Big(\max_{1\le j\le N} u_j^{\varepsilon} \Big) - \frac{1}{N} \sum_{j\ne m} \phi_\varepsilon \big(u_j^{\varepsilon}\big)$$

$$+ \frac{1}{N} \Big[\phi_\varepsilon \Big(\max_{1\le j\le N} u_j^{\varepsilon} \Big) - \phi_\varepsilon \big(u_m^{\varepsilon}\big) \Big]$$

$$\ge \frac{1}{N} \Big[\phi_\varepsilon \Big(\max_{1\le j\le N} u_j^{\varepsilon} \Big) - \phi_\varepsilon \big(\tfrac{1}{N}\big) \Big] \ge 0$$

where $u_m^{\varepsilon} = \max_{1\le j\le N} u_j^{\varepsilon}$ and we have used the fact that $\sum_{j=1}^{N} u_j^{\varepsilon} \equiv 1$.

Hence $\displaystyle\int_{\Omega} \Big[\sum \phi^{\varepsilon} - \phi_\varepsilon\big(\tfrac{1}{N}\big)\Big]_+^2 dx \;\le\; \int_{\Omega} \Big(\phi_\varepsilon \Big(\max_{1\le j\le N} u_j^{\varepsilon} \Big) - \phi_\varepsilon \big(\tfrac{1}{N}\big)\Big)^2$

$$\leq \quad N^2 \int_\Omega \max_{1 \leq j \leq N} \left(g_j^\varepsilon\right)^2 dx \leq N^2 K_5 t^{-1}$$

and this together with (2-26) yields

$$\int_\Omega \left(\sum \phi^\varepsilon - \phi_\varepsilon\left(\tfrac{1}{N}\right)\right)^2 dx \leq K_6 t^{-1} \tag{2-27}$$

Combining (2-24) and (2-27) we obtain

$$\sum_{i=1}^{N} \| \phi_\varepsilon (u_i^\varepsilon) \|^2 + \| \sum \phi^\varepsilon \|^2 \leq K_7 t^{-1} \tag{2-28}$$

which completes the proof of the proposition. □

§3 Proof of Theorem 1

It follows from the results of §2 that there exist $\{\mathbf{u}^\varepsilon, \mathbf{w}^\varepsilon\}$ uniformly bounded independently of ε in the spaces,

$$\mathbf{u}^\varepsilon \in C\left[0,T;\left(\mathbf{H}^1(\Omega)\right)'\right] \cap L^\infty\left(0,T;\ \mathbf{H}^1(\Omega)\right) \tag{3-1a}$$

$$\sqrt{t}\ d\mathbf{u}^\varepsilon/dt \in L^2\left(0,T;\ \mathbf{H}^1(\Omega)\right) \tag{3-1b}$$

$$\overline{\mathbf{w}}^\varepsilon = \mathbf{w}^\varepsilon - ⨍\ \mathbf{w}^\varepsilon \in L^2\left(0,T;\ \mathbf{H}^1(\Omega)\right) \tag{3-1c}$$

$$\sqrt{t}\ \mathbf{w}^\varepsilon \quad L^\infty\left(0,T;\ \mathbf{H}^1(\Omega)\right) \tag{3-1d}$$

such that

$$\sqrt{t}\ \boldsymbol{\phi}^\varepsilon(\mathbf{u}^\varepsilon) \in L^\infty\left(0,T;\mathbf{L}^2(\Omega)\right) \tag{3-2}$$

$$\mathbf{u}^\varepsilon(\cdot,0) = \mathbf{u}_0 \tag{3-3}$$

and for each $\xi \in C[0,T]$ and $\boldsymbol{\eta} \in \mathbf{H}^1(\Omega)$,

$$\int_0^T \xi(t)\left\{ \frac{d}{dt}\langle \mathbf{u}^\varepsilon, \eta\rangle + (\boldsymbol{L}\,\nabla\,\mathbf{w}^\varepsilon, \nabla\boldsymbol{\eta})\right\} dt = 0 \tag{3-4a}$$

$$\int_0^T \xi(t) \left\{ \left(\mathbf{w}^\varepsilon - \theta\, \boldsymbol{\phi}^\varepsilon(\mathbf{u}^\varepsilon) + \mathbf{A}\mathbf{u}^\varepsilon - \mathbf{e} \sum \left(\theta\, \boldsymbol{\phi}^\varepsilon(\mathbf{u}^\varepsilon) - \mathbf{A}\mathbf{u}^\varepsilon\right), \boldsymbol{\eta}\right) \right. \tag{3-4b}$$

$$\left. - \gamma\left(\nabla \mathbf{u}^\varepsilon, \nabla \boldsymbol{\eta}\right) \right\} dt = 0.$$

Thus passing to the limit $\varepsilon = 0$ in (3-4) using (3-1) and (3.2) yields a pair $\{\mathbf{u}, \mathbf{w}\}$ satisfying (1-20) provided we can show that

$$\lim_{\varepsilon \to 0} \int_0^T \xi(t) \left(\boldsymbol{\phi}^\varepsilon(\mathbf{u}^\varepsilon), \boldsymbol{\eta}\right) dt = \int_0^T \xi(t) \left(\boldsymbol{\phi}(\mathbf{u}), \boldsymbol{\eta}\right) dt \tag{3-5}$$

It follows from (2-9c) that $\mathbf{u} = \lim_{\varepsilon \to 0} \mathbf{u}^\varepsilon$ satisfies

$$\{\mathbf{u}\}_i \in [0,1] \qquad \forall i \tag{3-6}$$

and from (3-2) that there exists $\boldsymbol{\phi}^*$ such that

$$\sqrt{t}\, \boldsymbol{\phi}^* \in L^\infty\left(0, T; \mathbf{L}^2(\Omega)\right)$$

and

$$\lim_{\varepsilon \to 0} \int_0^T \xi(t) \left(\boldsymbol{\phi}^\varepsilon(\mathbf{u}^\varepsilon), \boldsymbol{\eta}\right) dt = \int_0^T \xi(t) \left(\boldsymbol{\phi}^*, \boldsymbol{\eta}\right) dt .$$

Hence in order to obtain (3-5) we have to show that

$$\{\phi^*\}_i = \phi(u_i). \tag{3-7}$$

Since (3.2) holds it follows that for each $M > 0$

$$t\left|\left[\left|\phi_\varepsilon(u_i^\varepsilon)\right| > M\right]\right| < \frac{c}{M^2}. \tag{3-8}$$

Set

$$F_M(v) := \max\left\{-M, \min\{M, v\}\right\}. \tag{3-9}$$

For each $\tau > 0$ it holds that, using (3-8),

$$\left|\int_\tau^T \xi(t)\left(\phi_\varepsilon(u_i^\varepsilon) - F_M\left(\phi_\varepsilon(u_i^\varepsilon)\right), \eta\right) dt\right|$$

$$\leq \left|\int_\tau^T \xi(t) \int_{[|\phi_\varepsilon(u_i^\varepsilon)| > M]} \left(|\phi_\varepsilon(u_i^\varepsilon)| + M\right)|\eta|\, dx\, dt\right|$$

$$\leq C(\tau)\, \|\xi\|_\infty \|\eta\|_\infty \Big/ M.$$

Since

$$\lim_{\varepsilon \to 0} F_M\left(\phi_\varepsilon(u_i^\varepsilon)\right) = \lim_{\varepsilon \to 0} F_M\left(\phi(u_i^\varepsilon)\right)$$

$$= F_M\left(\phi(u_i)\right)$$

it follows that the left hand side of the above inequality converges to

$$\left| \int_\tau^T \xi(t) \left(\boldsymbol{\phi}_i^* - F_M\left(\phi(u_i)\right), \eta \right) dt \right| < \frac{c(\tau) \|\xi\|_\infty \|\eta\|_\infty}{M}$$

Taking $\eta = F_M(\phi(u_i))$ we find that

$$\int_\tau^T \| F_M(\phi(u_i)) \|^2 \, dt \le C(\tau) \ \forall \ M$$

which implies

$$\int_\tau^T \| \phi(u_i)) \|^2 \, dt \le C(\tau)$$

and

$$\boldsymbol{\phi}_i^* = \phi(u_i) \quad \text{on} \quad (\tau, T) \ .$$

This completes the proof of (3-7) since τ is arbitrary.

In order to prove uniqueness we use the idea given in Blowey and Elliott [1991a]. Let $\mathbf{f} = \{f_i\}_{i=1}^N$ where

$$f_i \in \left(H^1(\Omega)\right)' \quad , \quad \langle f_i \, , \, 1 \rangle = 0 \quad ; \quad \sum_{i=1}^{N} f_i = 0 \; . \tag{3.10}$$

We introduce the Green's operator G defined by: -

$$G\,\mathbf{f} \in \mathbf{H}^1(\Omega) \; , \quad \sum G\,\mathbf{f} = 0 \; , \quad \fint G\,\mathbf{f} = 0 \tag{3-11a}$$

$$\left(\boldsymbol{L}\,\nabla G\,\mathbf{f} \, , \, \nabla \boldsymbol{\eta}\right) = \langle \mathbf{f} \, , \, \boldsymbol{\eta} \rangle \qquad \forall\, \boldsymbol{\eta} \in H^1(\Omega) \tag{3-11b}$$

That (3-11) defines a unique $G\,\mathbf{f}$ for an $\mathbf{f}$ satisfying (3-10) follows from (1-15) and the Lax-Milgram theorem.

Let $\left\{ \mathbf{z}^u , \, \mathbf{z}^w \right\} = \left\{ \mathbf{u}^1 - \mathbf{u}^2 \; , \; \mathbf{w}^1 - \mathbf{w}^2 \right\}$ be the difference of two pairs of solutions to (1-20). Using the monotoncity of $\phi(\cdot)$ we find from (1-20b) that

$$\gamma \, \| \nabla \mathbf{z}^u \|^2 \le \left(\mathbf{z}^u , \, \mathbf{z}^w \right) + \lambda_A \| \mathbf{z}^u \|^2 \; .$$

Since, by (1-20a),

$$\mathbf{z}^w = - G\, \mathbf{z}^u{}_t$$

it follows that

$$\frac{1}{2} \, \frac{d}{dt} \, \| \boldsymbol{L} \, \nabla \, G \, \mathbf{z}^u \|^2 + \gamma \, \| \nabla \mathbf{z}^u \|^2 \le \lambda_A \left(\boldsymbol{L} \, \nabla \, G \, \mathbf{z}^u \; , \; \nabla \mathbf{z}^u \right) .$$

A standard Gronwall argument yields uniqueness since

$$\mathbf{z}^u(0) = 0 .$$

§4 Proof of Theorem 2

Denoting by $\{\mathbf{u}^{\theta}, \mathbf{w}^{\theta}\}$ the solution of (1-20) for fixed θ, it is clear that from the estimation given in the proof of Theorem 1 that we may pass to the limit.

$$\{\mathbf{u}, \mathbf{w}\} = \lim_{\theta \to 0} \{\mathbf{u}^{\theta}, \mathbf{w}^{\theta}\}$$

and we need only justify the variational inequality (1-21b) and the uniqueness of the limit.

Let $\eta^{\alpha} \in K^{+}$ and $\eta^{\alpha} \geq \alpha\,\mathbf{e}$ for some small positive α. Since $\sum\left(\eta^{\alpha} - \mathbf{u}^{\theta}\right) = 0$ we have

$$0 = \left(\eta^{\alpha} - \mathbf{u}^{\theta}, \mathbf{e}\sum \mathbf{v}\right) \qquad \forall\ \mathbf{v} \in \mathbf{L}^{2}(\Omega)\ .$$

Furthermore $\phi\left(\eta^{\alpha}\right) \in \mathbf{L}^{2}(\Omega)$ because $\eta^{\alpha} \geq \alpha\,\mathbf{e}$. Hence it follows from (1-20b) and the monotoncity of $\phi(\cdot)$ that for for $\xi(\geq 0) \in C[0,T]$,

$$\int_{0}^{T} \xi(t)\left\{\gamma\left(\nabla \mathbf{u}^{\theta}, \nabla \eta^{\alpha}\right) - \left(\mathbf{w}^{\theta} + A\mathbf{u}^{\theta}, \eta^{\alpha} - \mathbf{u}^{\theta}\right)\right\} dt$$

$$= \int_{0}^{T} \xi(t)\,\gamma\left(\nabla \mathbf{u}^{\theta}, \nabla \mathbf{u}^{\theta}\right) dt$$

$$+ \int_{0}^{T} \xi(t)\left\{\theta\left(\phi\left(\eta^{\alpha}\right) - \phi\left(\mathbf{u}^{\theta}\right), \eta^{\alpha} - \mathbf{u}^{\theta}\right)\right\} dt$$

$$- \int_{0}^{T} \xi(t)\,\theta\left(\phi\left(\eta^{\alpha}\right), \eta^{\alpha} - \mathbf{u}^{\theta}\right) dt$$

$$\geq \int_0^T \xi(t)\, \gamma\left(\nabla \mathbf{u}^\theta, \nabla \mathbf{u}^\theta\right) dt \quad - \quad \int_0^T \xi(t)\, \theta\left(\phi\left(\eta^\alpha\right), \eta^\alpha - \mathbf{u}^\theta\right) dt.$$

By the weak and strong convergence properties of $\{\mathbf{u}^\theta, \mathbf{w}^\theta\}$ as $\theta \to 0$ we may pass to the limit and obtain,

$$\int_0^T \xi(t) \left\{\gamma\left(\nabla \mathbf{u}, \nabla \eta^\alpha\right) \; - \; \left(\mathbf{w} + A\mathbf{u}, \; \eta^\alpha - \mathbf{u}\right) dt\right.$$

$$= \lim_{\theta \to 0} \int_0^T \xi(t) \left\{\gamma\left(\nabla \mathbf{u}^\theta, \; \nabla \eta^\alpha\right) \; - \; \left(\mathbf{w}^\theta + A\mathbf{u}^\theta, \; \eta^\alpha - \mathbf{u}^\theta\right)\right\} dt$$

$$\geq \liminf_{\theta \to 0} \int_0^T \xi(t)\, \gamma\left(\nabla \mathbf{u}^\theta, \; \nabla \mathbf{u}^\theta\right) dt \; - \; \lim_{\theta \to 0} \int_0^T \xi(t)\, \theta\left(\phi\left(\eta^\alpha\right), \eta^\alpha - \mathbf{u}^\theta\right) dt$$

$$\geq \int_0^T \xi(t)\, \gamma\left(\nabla \mathbf{u}, \; \nabla \mathbf{u}\right) dt.$$

Furthermore, since any $\eta \in K^+$ can be approximated by $\eta^\alpha \in K^+$. For small α with $\eta^\alpha \geq \alpha\, \mathbf{e}$, we may pass to the limit $\alpha = 0$ in the left hand side of the above inequality and obtain (1-21b).

Uniqueness is proved in the same way as for the $\theta > 0$ problem.

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